Key Terms and Components of Dynamic Systems:

Dynamic Systems: Systems that contain *time* as a parameter; such systems "evolve" over time.

State Variable: A state variable describes the status, or "state of being," of one of the variables in the system.

Initial Conditions: Values that the state variables take on at the beginning of the time period of interest.

Control Variable: A control variable is a variable that is under the control of some individual or group.

Random variables (noise variables): Uncontrolled variables which can assume several values with certain probabilities.

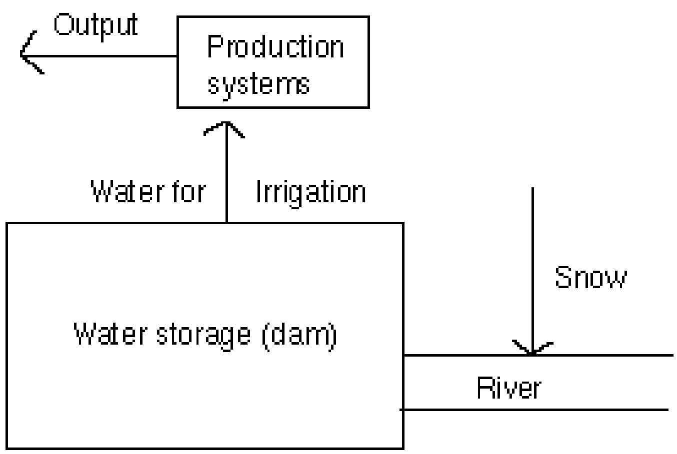
Constraints: Equations (or inequalities) which limit the values that state variables or control variables can take on.

Equation of Motion: The equation of motion describes how a variable changes over time.

Solution of a System: The solution of a dynamic system is a set of *equations*, where the equations are in terms of the system parameters, *including time*, such that all of the original equations in the system are satisfied. Thus, a dynamic system may have many solutions, depending on the specific initial conditions of the resource.

Objective Function: An objective function is an equation that measures how well the system is attaining some goal or objective, usually expressed in terms of the *state variables, control variables,* and *parameters* of the system.

Example: Set-up for Natural Resource Dynamic System



State variables:

(S_t): Denotes the level of a stock at time t; (e.g., the quantity of water stored in the reservoir behind the dam at time t).
(U_t): Uncontrolled inputs, (e.g., rain, snow).
(Y_t): Outputs; outcome of systems at time t; (e.g., crops produced)

Control variables:

(X_t): Inputs whose magnitudes we can choose in our attempt to reach our objectives. (e.g., the amt. of water used for irrigation).

Parameters:

(P): Items that can be taken as constant with respect to the problem at hand. (e.g., the production elasticity of irrigated water)

Equation of motion:

Next period water stock = This period water stock + rainfall - irrigation water:

$$S_{t+1} = S_t + U_t - X_t$$

Objective Function:

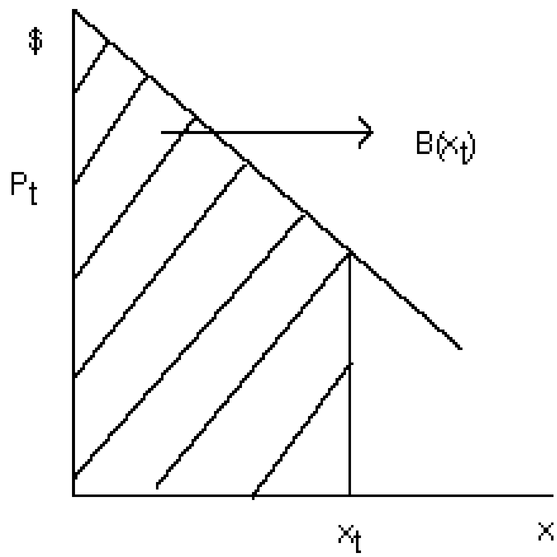
$$\max_{X_t} NPV = \sum_{t=0}^{T} \frac{B_t[Y_t(P)] - C_t(X_t)}{(1+r)^t}$$

Dynamic Models of Nonrenewable Resources

Nonrenewable resources are resources that have a finite stock and that do not grow naturally.

Key Issues:

- Determining optimal resource allocation and pricing.
- Sources of market failure and policies to correct market failure.



t = time (the initial period: t=0; the future period: t=1)

r = interest rate

S_0 = initial stock of nonrenewable resource

X_t =control variable, the amt. of the resource consumed in period t

$B(X_t)$ = benefit of consuming X_t

Economics of scarcity:

- **Scarcity:** Imposes an opportunity cost on using resources today. In a natural resource system, we refer to dynamic opportunity cost as a *user cost.*
- **User Cost:** The Present Value of foregone opportunity. (e.g., if you use a unit of a natural resource today, you forego the opportunity to use it tomorrow)

Nonrenewable Resources (cont.)

The User Cost decreases as r increases:

- The higher the interest rate, the less valuable tomorrow's benefits and the smaller the opportunity cost of using more of the resource today.
- at r = infinity, resources left for tomorrow are worth nothing and user cost = 0.
- Similarly, when there is enough of the resource to go around, so that scarcity is not an issue, the user cost = 0. The dynamic model yields the same outcome as two separate static models.

Discounting: The use of discounting is important in determining the optimal extraction rate of a nonrenewable resource, because the revenue a resource owner receives in period 1 is not worth as much as the revenue received in period 0.

- the NPV of benefits in period 1 in terms of the current period 0:

$$\text{NPV} = \left(\frac{1}{1+r}\right) B(X_1)$$

Dynamic Efficiency: An allocation of resources is said to be **dynamically efficient** when it maximizes the NPV of benefits.

$$\text{Max. } L = B(X) - C(X),$$

- B(X) is now a *stream of benefits* through time,

$$B(X) = B_0 + \left(\frac{1}{1+r}\right)B_1 + \left(\frac{1}{1+r}\right)^2 B_2 + \ldots + \left(\frac{1}{1+r}\right)^N B_N$$

- C(X) is now a *stream of costs* through time

Dynamic Efficiency: The Two Period Case

assume zero costs are associated with consuming the resource.

Objective function: $\underset{X_0, X_1}{\text{Max}} \ NPV = B(X_0) + \frac{1}{1+r} B(X_1).$

Equation of motion (constraint): $S_0 = X_0 + X_1$.

Note: by assuming $X_0 + X_1$ *exactly* equals S_0 (resource stock is used up), we are implicitly assuming **unsatiated demand**.

the optimization problem is:

$$\underset{X_0, X_1}{\text{Max}} \ NPV = B(X_0) + \frac{1}{1+r} B(X_1)$$

subject to: $S_0 = X_0 + X_1$..

The Lagrangian expression is:

$$L = B(X_0) + \frac{1}{1+r} B(X_1) + \lambda \left(S_0 - X_1 - X_0\right).$$

To maximize the Lagrangian expression we find the F.O.C.'s:

(1) $$\frac{dL}{dX_0} = B_x(X_0) - \lambda = 0$$

(2) $$\frac{dL}{dX_1} = B_x(X_1)\left(\frac{1}{1+r}\right) - \lambda = 0$$

(3) $$\frac{dL}{d\lambda} = S_0 - X_1 - X_0 = 0$$

Two-period Dynamic Efficency (cont.)

The system can be solved for X_0, X_1 and λ in terms of the parameters of the system. An often useful step in this process is to set FOC (1) = FOC (2) and eliminate λ to obtain:

$$(4) \quad B_X\left(X_0\right) = \frac{1}{1+r} B_X\left(X_1\right)$$

- then use (3) and (4) to solve for X_0 and X_1, and
- substitute X_0 into (1) to find λ.

We can find P_0 and P_1 by recalling that:

(5) $B_X(X_t)$ = MB of X at time t = Price at time t = P_t

Rearranging (4), we get: $$(1+r)\ B_X(X_0) = B_X(X_1)$$

Substituting P_0 for $B_X(X_0)$ and P_1 for $B_X(X_1)$, we find:

$$\frac{P_1 - P_0}{P_0} = r$$

Two-period Dynamic Efficency (cont.)

Conclusions:

- when dynamic efficiency is met, the price increases at the rate of interest.

- the shadow price of S_0, λ , is equal to P_0. the shadow value is also equal to the *present value of* P_1. In other words, $\lambda = P_0 = P_1/(1+r)$. Thus, the solution to the nonrenewable resource problem equates the NPV of benefits across all time periods in the horizon

- If $P_0 > P_1/(1+r)$, the owner should extract more today; invest the money at r.

- If $P_0 < P_1/(1+r)$, the owner should leave more in the ground to extract tomorrow

- the rate of return of holding resource stock in the ground is: IRR > r.

- Therefore, in equilibrium, it must be the case that $P_0 = P_1/(1+r)$.

  -Produce today until $MB_0 = PV(MB_1)$

Note: The intuition for λ is that, **λ = the user cost of the resource!** The solution to the dynamic problem equates the user cost of extracting the resource across all time periods.

A Numerical example:

Suppose $B(X) = a\sqrt{X}$

then $B_X(X) = \dfrac{a}{2\sqrt{x}}$.

noting that X1 = S0 - X0 from (3),

X_0 can be found by using $B_X(X)$ with eqn's (3) and (4) :

$$\frac{a}{2\sqrt{X_0}} = \frac{a}{2(1+r)\sqrt{S_0 - X_0}} \qquad \frac{S_0 - X_0}{X_0} = \frac{1}{(1+r)^2}$$

$$(6) \qquad X_0 = S_0 \frac{(1+r)^2}{1+(1+r)^2}$$

Substitute X_0 back into eqn (3) to find X_1 :

$$(7) \qquad X_1 = \frac{S_0}{1+(1+r)^2}$$

Substitute X_0 back into $B_X(X_0)$ to find :

$$(8) \qquad P_0 = \frac{a}{2}\sqrt{\frac{1+(1+r)^2}{S_0(1+r)^2}}$$

If S_0 increases, then both X_0 and X_1 increase

if r increases, then X_0 increases & X_1 decreases and P_0 decreases.

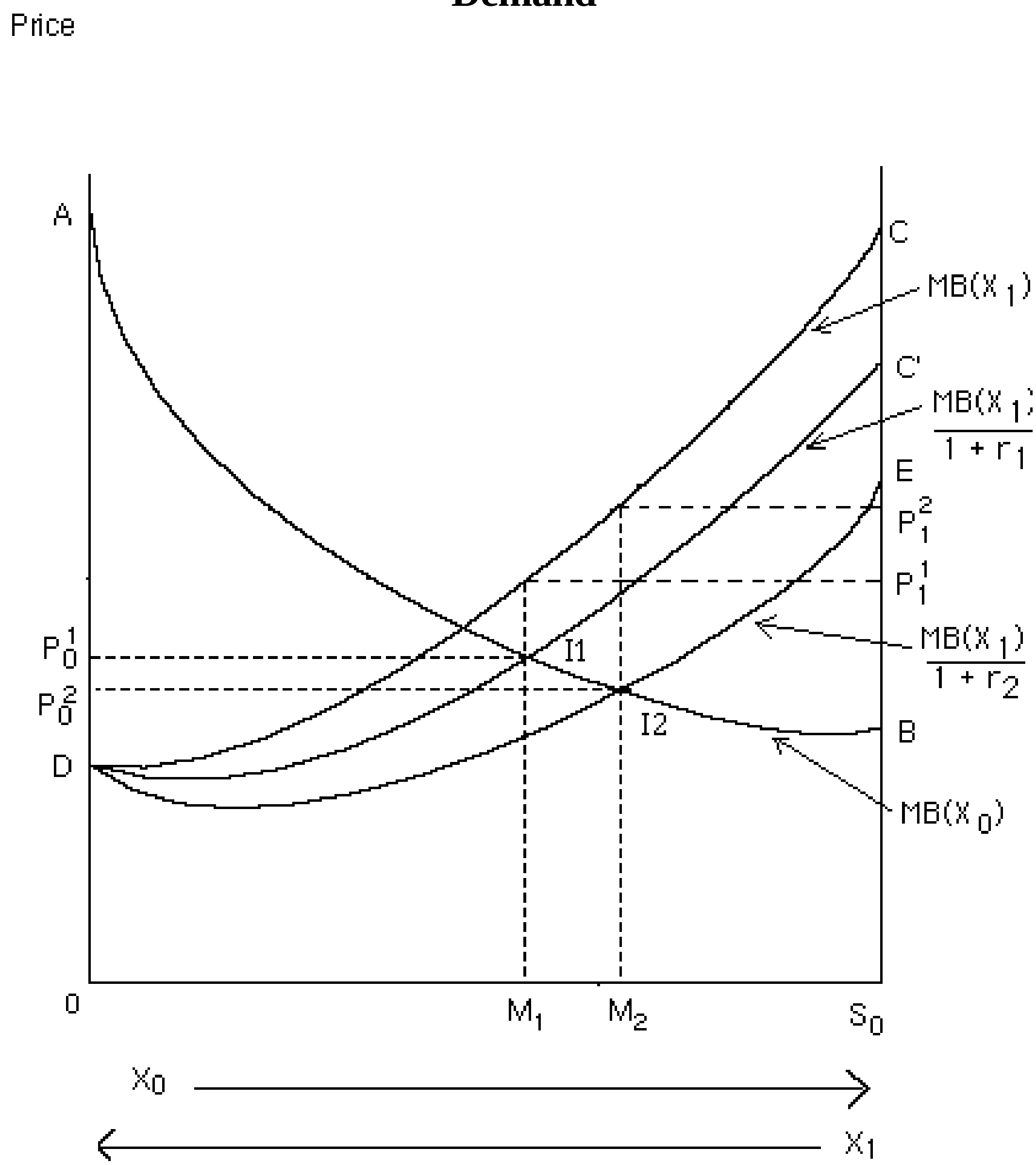
- if r = 0.1, S_0 = 100 and a = 10, then:

X_0 = 54.75, X_1 = 45.25, P_0 = 0.68 and P_1 = 0.74

- If r increases to r = 0.5, then:

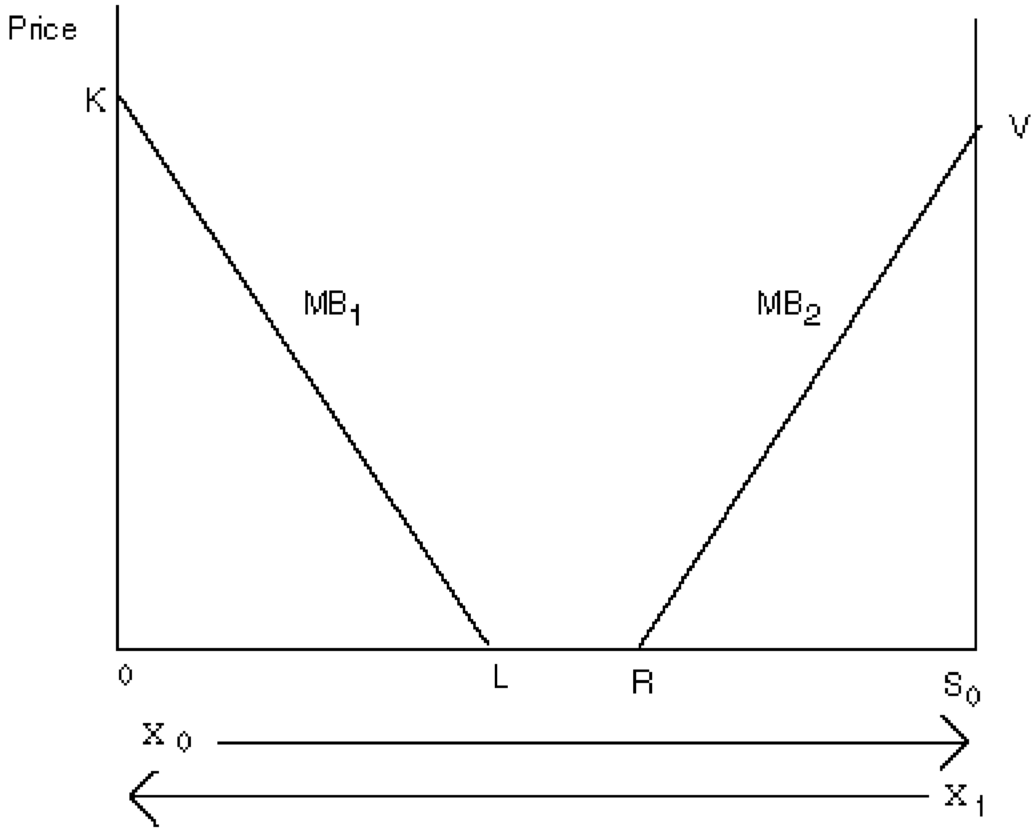
X_0 = 69.3, X_1 = 31.7, P_0 = 0.6 and P_1

Two-Period Non-renewable Resource Model with Unsatiated Demand



- For P's: superscript = discount rate; subscripts = time period.
- For I's, M's, subscripts = discount rate.
- $r_2 > r_1$; $I_1 < I_2$.
- A lower discount rate implies:
  - i) $P_0^1 > P_0^2$ Higher price in the initial period.
  - ii) $P_1^1 < P_1^2$ Lower price in the second period.
  - iii) $M_1 < M_2$ Less resource is used in the initial period.

Two-Period Non-renewable Resource Model with Satiated Demand



When S_0 is so large that B_{X_0} and $\frac{1}{1+r} B_{X_1}$ do not intersect at positive P, then:

- X_0 is solved for by setting $B_X(X_0) = 0$, and
- X_1 is solved for by setting $B_X(X_1) = 0$

This solution is identical to the solution of two individual static maximization problems, performed separately, in period 0 and period 1.

Note: There is no user cost here, because the MB curves fail to intersect. That is, there is no scarcity in a nonrenewable resource model with satiated demand.