Suggested Solutions to Problem Set 4*

1. Assume you receive a flow of income (V) at the end of every year for (N) years. This income grows at a rate of (G) percent per year, and the discount factor is given by the nominal interest rate (I). The present value (PV) of this flow of income is given by

$$PV = \sum_{j=1}^{N} \frac{V(1+G)^{j-1}}{(1+I)^{j}} \qquad (1)$$

Let's multiply both sides of equation (1) by $(1+G)$

$$\begin{aligned} PV\,(1+G) &= \sum_{j=1}^{N} V\left(\frac{1+G}{1+I}\right)^{j} \\ &= \sum_{j=1}^{N} V\beta^{j}, \text{ where } \beta = \left(\frac{1+G}{1+I}\right) \qquad (2) \end{aligned}$$

Note that this geometric series converge if and only if $\beta < 1$, which means that $G < I$

Let's multiply again both sides of equation (2) by β

$$PV\,(1+G)\beta = \sum_{j=1}^{N} V\beta^{j+1} \qquad (3)$$

Subtracting equation (3) from equation (2), we obtain

$$PV\,(1+G)(1-\beta) = V\beta(1-\beta^{N}) \qquad (4)$$

Upon dividing both sides of (4) by $(1+G)(1\text{-}\beta)$, and after substituting back the value of β, we finally obtain

$$\frac{V}{I\text{-}G}\left[1-\left(\frac{1+G}{1+I}\right)^{N}\right] \qquad (5)$$

Note that if $N \to \infty$, then $PV = \dfrac{V}{I-G}$.

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* Solution to question 1 provided by G. Malick. Solution to question 2 provided by S. Marceau.

The nominal interest rate (I) is approximately equal to the sum of the real interest rate (r) and the inflation rate (π)[1]

$$I = r + \pi \qquad (6)$$

Similarly, the growth in the value of income (G) is explained by both the biological growth in the volume of timber (Gb) and the growth in the price of timber (Gp)[2].

$$G = Gb + Gp \qquad (7)$$

Substituting (6) and (7) into (5)

$$PV = \frac{V}{(r+\pi) - (Gb + Gp)}\left[1 - \left(\frac{1 + Gb + Gp}{1 + r + \pi}\right)^N\right] \qquad (8)$$

With equation (8) in hand, we can now proceed to answer the question.

(a) N = 10,000/400 = 25 ; V = 400P; r =3%; π = 3%; Gb = 2%; Gp = 0. Hence, the present value (or what the government should pay to prevent cutting) is given by

$$PV = \frac{400P}{.04}\left[1 - \left(\frac{1+.02}{1+.06}\right)^{25}\right] = \$6,177\ P$$

(b) A year has passed (so we have 24 years left) and the inflation rate is now only 2%. N = 24; V = 400P; r = 3%; Gb = 2%; Gp = 0 and π = 2%. The present value as of that moment (a year later) is

$$PV = \frac{400P}{.03}\left[1 - \left(\frac{1+.02}{1+.05}\right)^{24}\right] = \$6,684\ P$$

A reduction in the inflation rate reduces the nominal interest rate, so the present value of the flow of income increases (the government needs to pay him a little bit more to prevent him from cutting).

(c) N = 25; V = 400P; r=3%; π = 3%; Gb = 2%; but Gp = 1%. The present value is

[1] This is not entirely correct. The nominal interest rate is given by $I = r + \pi + r\pi$. Since $r\pi$ is very small for a low real interest rate and for a low inflation rate, then we sometimes ignore this.
[2] Again, this is not entirely correct. The growth in value is given by G = Gb +Gp + GbGp, so for small levels of growth we sometimes ignore GbGp.

$$PV = \frac{400P}{.03}\left[1-\left(\frac{1+.03}{1+.06}\right)^{25}\right] = \$6{,}829\ P$$

The expected increase of 1% per year in the price of timber will increase the present value of the flow of income. Again, cutting would be more profitable and the government would have to pay more to prevent it.

2. This problem required that you calculate the extraction levels and prices for a non-renewable resource under various scenarios. The table gives an overview of the results.

variable	open access	backstop	optimal	monopoly
x_0	2200	1611.76	1506.93	1100
x_1	800	1388.36	1493.07	1100
z_1	na	21.64	na	na
p_0	150	297.06	323.27	425
p_1	500	300	326.73	425

(a) If the producers are engaging in anti-competitive behavior, then the monopoly price should be close to the observed price. To determine this, we need to find the monopoly price. A monopolist will set its marginal profit from selling one more unit in the first period equal to its marginal profit from selling that unit in the second period instead. This implies marginal revenue minus marginal cost in period 0 will be equal to the present value of marginal revenues minus marginal costs in period 1. That is,

$$MR(x_0^m) - MC(x_0^m) = \frac{MR(x_1^m) - MC(x_1^m)}{1+r} \qquad (1)$$

Marginal revenue is the first derivative of total revenue with respect to x. This is:

$$TR(x) = p(x)\cdot x = (700-0.25x)x = 700x - 0.25x^2 \Rightarrow MR = \frac{\partial TR}{\partial x} = 700 - 0.5x.$$

The marginal profit at any point in time is therefore:

$$MR(x) - MC(x) = 700 - 0.5x - 150 = 550 - 0.5x.$$

Equation 1 becomes $550 - 0.5x_0^m = \frac{550-0.5x_1^m}{1+0.02} = \frac{550-0.5(3000-x_0^m)}{1.02}$. Rearranging gives $1.02(550-0.5x_0^m) = -950 + 0.5x_0^m \Rightarrow x_0^m = \frac{561+950}{1.01} = 1496.04$ and x_1^m = 3000-1496.04 = 1503.96.

To get this answer, I have assumed the monopolist uses all of the resource. This assumption needs to be checked before proceeding. The user cost is

$l^m = \frac{MR(x_1^m) - MC(x_1^m)}{1+r} = \frac{550 - 0.5 \cdot 1503.96}{1.01} = -199.98.5 < 0$. The monopolist's net marginal profits are negative, i.e., it could make more money by reducing its output. It will not extract all of the resource and the relevant condition is to set net marginal profit equal to zero. This also implies that the monopolist will produce the same amount in both periods: $MR(x^m) - MC(x^m) = 0 \Rightarrow 550 - 0.5x = 0 \ and \ x_0^m = x_1^m = 1100$. Prices will also be the same in both periods: $p_0^m = p_1^m = 700 - 0.25 \times 1100 = 425$. The extraction level and price in each period are much higher than the observed price of \$297.06. For a monopolist, we have found:

$$x_0^m = 1100 \ tons, \ x_1^m = 1100 \ tons$$
$$p_0^m = \$425/ton, \ p_1^m = \$425/ton$$

(b) Under open access, profits are driven to zero. That is, $\Pi^{oa} = px - cx = 0 \Rightarrow p^{oa} = c$. The outcome would be $p_0^{oa} = 150 \ and \ 700 - 0.25x_0^{oa} = 150 \Rightarrow x_0^{oa} = \frac{550}{0.25} = 2200$ in the first period and $x_1^{oa} = 3000 - 2200 = 800 \ and \ p_1^{oa} = 700 - 0.25 \times 800 = 500$ in the second period. The observed price in the current period greatly exceeds the price we would observe if there was an open access problem (\$297.06 > \$150). To summarize:

$$x_0^{oa} = 2200 \ and \ x_1^{oa} = 800$$
$$p_0^{oa} = 150 \ and \ p_1^{oa} = 500$$

(c) The BLM could sell off its lands in an auction. Each buyer would then have entitlement to the land he bought, i.e., would have perfect property rights. However, the BLM says it solved the problem **on public lands**. Open access is similar to an externality problem: agents making decisions in this particular market are not taking into account the full costs of their production decisions. The solution here is the same as it was when pollution was the externality: taxes equal to the unaccounted portion of marginal costs. This unaccounted portion is the user cost λ. If the BLM sets the tax equal to the user cost that prevails under the optimal solution, then t = λ*. When the miners drive profits to zero now, they end up with p = c + λ* = 150 + 173.27 = 323.27, which is the optimal price.

(d) The socially optimal extraction must satisfy the condition that net marginal benefits in period 0 equal the present value of net marginal benefits in period 1:

$$MB(x_1^*) - MC(x_0^*) = \frac{MB(x_0^*) - MC(x_1^*)}{1+r} \qquad (2)$$

For our specific problem, this becomes:

$700-0.25x_0^*-150=\frac{700-0.25x_1^*-150}{1.02} \Leftrightarrow 550-0.25x_0^*=\frac{550-0.25(3000-x_0^*)}{1.02}$, where I have used the assumption that $x_0^*+x_1^*=3000$. Solving for x_0^* gives $x_0^*=\frac{761}{0.505}=1506.93$ and x_1^* = 1493.07.

We should verify our assumption that all of the resource is extracted by calculating the user cost: $l^*=\frac{MB(x_1^*)-MC(x_1^*)}{1+r}=\frac{700-0.25\times 1493.07-150}{1.02}=173.27>0$. All of the resource is indeed extracted. After calculating the optimal prices, we notice that they are larger than the observed price, though closer than either the monopoly or the open access prices: $p_0^*=700-0.25\times 1506.93=323.27$ and $p_1^*=700-0.25\times 1493.07=326.73$. To summarize, the equilibrium is:

$$x_0^*=1506.93 \textit{ and } x_1^*=1493.07$$
$$p_0^*=323.27 \textit{ and } p_1^*=326.73$$

(e) The aluminum in this exercise is a more expensive alternative to producing cans. It acts like a backstop: when we run out of tin, we can use aluminum. In this framework, tin is not really the good from which we derive benefits. We derive benefits from the material that allows us to produce cans. That material can be either tin or aluminum. If we continue to denote extraction of tin by *x*, then marginal benefits from consumption of the material will be *MB = p = 700 - 0.25(x+z)*, where *z* is the amount of aluminum extracted.

Next, we need to determine whether aluminum will be used in he first period. The lowest price an aluminum producer will accept to supply the product is $300/ton. The quantity demanded at that price is 1600 tons. Tin producers are able to supply the entire market. Rather than share the market with the aluminum producers, they can offer to sell the material at a price just below $300[3] and still make profit. At such a price, however, aluminum producers will stay out of the market. All this implies is that we can use the model seen in class of the allocation of a non-renewable resource when a backstop becomes available in the second period only.

Even though no aluminum is produced in the first period, its presence will affect extraction of tin and prices in both periods. When deciding how much tin to extract, the tin producer weighs the extra profit from extracting today with the extra profit from extracting tomorrow. The backstop affects expected marginal profit in the second period, effectively lowering it. In the absence of the backstop, the market price in the second period would be $326.73. This price exceeds the marginal cost of supplying the material because tin is scarce. If tin

[3] They aren't behaving collusively. This is a market-determined outcome. As we shall see, the price in the first period will indeed be shy of $300.

producers could supply more, they would because the each make more profit by doing so since their increase in revenues ($326.73) exceeds their increase in costs from supplying one more unit ($150), i.e., their profits would increase. When aluminum becomes available, however, that extra supply will come about. Producers will supply the material, aluminum in this case, as long as the price covers the cost of bringing the last unit of the material to market. Price will not exceed that marginal cost because aluminum, unlike tin, is not scarce. If anybody tries to sell at a higher price, someone else will bid the price down. Price would not fall below $300 either. If it did, only the lower cost tin producers would be willing to the supply the material. Yet, at a price below $300, the market demand would outstrip what the tin producers are capable of supplying. There would be excess demand and we do not have an equilibrium. Thus, price in the second period will be equal to the marginal cost of the backstop.

Now that we have this, we can calculate the user cost, which the tin producer will compare with net marginal benefits in the first period when deciding how much to extract. Let *m* denote the marginal cost of mining aluminum, and *c* denote the marginal cost of mining tin, and superscripts *b* refer to the case when there is a backstop, the user cost is

$$l^b = \frac{MB(x_1^b + z_1) - MC(x_0^b)}{1+r} = \frac{p_1^b - c}{1+r} = \frac{m-c}{1+r} = \frac{300-150}{1.02} = 147.06$$

Extraction of tin in the first period continues until the net marginal benefits are equal to the user cost: $MB(x_0^b) - MC(x_0^b) = l^b \Rightarrow 700 - 0.25x_0^b - 150 = 147.06$. Solving this equation implies extraction in the first period of $x_0^b = \frac{402.94}{0.25} = 1611.76$ and extraction in the second period of $x_1^b = 3000 - 1611.64 = 1388.36$. By derivation, the user cost is positive, so we know that all of the tin will be extracted over the two periods.

Price in the first period will be $p_0^b = 700 - 0.25 \times 1611.76 = 297.06$. The puzzle is solved! We are observing a socially efficient market with an alternative source of materials waiting in the wings for the price to rise.

We can also find the remaining unknown variable in the equation: the level of production from the backstop. At the second period price, the quantity of materials demanded is given by $p_1^b = 300 = 700 - 0.25(1388.36 + z_1) \Rightarrow z_1 = \frac{52.91}{0.25} = 211.64$. To summarize, we have found that with a backstop technology:

$$y_0^b = x_0^b = 1611.76 \textit{ and } x_1^b = 1388.36$$
$$p_0^b = 297.06 \textit{ and } p_1^b = 300$$
$$z_1 = 211.64 \textit{ and } y_1^b = x_1^b + z_1 = 1600$$