EEP101/ECON125 Spring 00
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PROBLEM SET 5 Solutions

1. *Suppose r=0.1, price (P) of timber is \$10 per boardfoot, and the volume of trees in a stand obeys the function* $Q(T) = 12\,T^2 - 1/3\,T^3$.

   a) *Consider a single rotation. Set up the profit maximization problem and derive the equilibrium condition. Solve for the optimal rotation length (T*). Be sure to check to see if your answer makes sense.*

The problem assumed no harvesting costs. The grower's objective is to choose the rotation length to maximize profits (or revenues in this case). Of course, since this profit is realized T years from now, we need to discount this future profit to the present. This objective is expressed as

$$\underset{T}{MAX}\ P.V.\ \Pi = PQ(T)\,e^{-rT}$$

Maximizing this objective with respect to T gives one first order condition which implicitly defines T*:

$$FOC: \frac{d\Pi}{dT} = 0\ ?\quad PQ'(T^*)e^{-rT^*} - rPQ(T^*)e^{-rT^*} = 0$$

$$?\quad PQ'(T^*)e^{-rT^*} = rPQ(T^*)e^{-rT^*}$$

Since e^{-rt} and P cancel, this leaves us with

$$Q'(T^*) = rQ(T^*)$$

or,

$$\frac{Q'(T)}{Q(T)} = r \qquad (1)$$

This says that the profit maximizing rotation length is such that the growth of the tree volume is equal to the interest rate. Notice that Q'(T) refers to change in growth from one period to the next and represents the change in volume of trees. The growth rate of trees multiplied by 100 is the percentage growth of trees from one period to the next. Using the information given in the description of the problem, and $Q'(T) = 24T - T^2$, equation 1 is rewritten as

$$\frac{24T - T^2}{12\,T^2 - 1/3\,T^3} = 0.1 \qquad (2)$$

To find T*, we need to solve equation (2) for T. After factoring out a T and rearranging terms, equation (2) can be rewritten as

$$24 - T - 0.1(12T - \tfrac{1}{3}T^2) = 0\ ?\quad \tfrac{1}{30}T^2 - 2.2T + 24 = 0\ ?\quad T^2 - 66T + 720 = 0$$

We can solve this expression for T using the quadratic formula (Refresher: $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ where a,b, and c correspond to the coefficient of the first, second and third term of the quadratic function respectively). For our case, the value(s) of T for which equation (3) holds are found to be:

$$\frac{66 \pm \sqrt{66^2 - 4(720)}}{2} = \frac{66 \pm 38.42}{2}$$

which reduces to T=13.785 or 52.215.

This doesn't necessarily mean that rotation lengths of either 13.785 years or 52.215 years will maximize profits. Check both answers by plugging each choice back into the profit function $P*Q(T)e^{--rT}$

T=13.785: $P*Q(T)\, e^{-rT}$ $= \$10\, [12(13.785)^2 – 1/3\, (13.785)^3]e^{--(0.1)(13.785)}$
$= \$10(2280.3147 – 873.171)(0.252)$
= **\$3546.00**
T=52.215: $P*Q(T)\, e^{-rT}$ $= \$10\, [12(52.215)^2 – 1/3\, (52.215)^3]e^{--(0.1)(52.215)}$
$= \$10(32716.8747 – 47453.1)(0.0054)$
= **- \$795.76**

Actually what the T=52.215 is saying is that by waiting for about 52 years until harvesting these trees will result yield in a negative yield (Q(T=52.215)= - 14736.2, which doesn't make any sense[1]. Therefore, the rotation length that maximizes profits is 13.785 years.

b) *Consider an infinite series of rotation. Again set up the profit maximization problem and derive the equilibrium condition. (Note that you are not required to solve for T* for this case.)*

In the infinite rotation problem, the grower chooses the rotation length which maximizes the present value of all future profits. That is to say, he/she chooses T*, which maximizes the present value of profits from the first rotation, second rotation and so on. Since the profits are realized at the end of each rotation, they each need to be discounted to the present.

$$\underset{T}{MAX}\ P.V.\,\Pi = PQ(T)e^{-rT} + PQ(T)e^{-2rT} + PQ(T)e^{-3rT} + \bullet\bullet\bullet$$

The first term refers to the profit at the end of the first rotation, the second term, the profit at the end of the second rotation, etc.

To make things easier to work with (unless you choose to take the derivative of infinite terms with respect to T!), we need to rearrange this expression. We can to this by first factoring out the $PQ(T)e^{-rT}$ from the above expression.

$$\begin{aligned} &PQ(T)e^{-rT} + PQ(T)e^{-2rT} + PQ(T)e^{-3rT} + \bullet\bullet\bullet \\ &\qquad = PQ(T)e^{-rT}\left[1 + e^{-rT} + e^{-2rT} + e^{-3rT} + \bullet\bullet\bullet\right] \end{aligned}$$

[1] The negative yield result comes from the mathematical representation of the yield function, Q(T), which resulted in yield becoming negative for some number of years. This served to simplify solving for T*. Of course, this yield function doesn't make sense in reality since we can't have negative yields.

The term in the bracket is a sum of an infinite series[2] and so this entire expression can be rewritten as

$$PQ(T)\,e^{-rT}\,\frac{1}{1-e^{-rT}} = PQ(T)\,\frac{1}{e^{rT}-1}$$

The last term comes from multiplying the denominator of the left-hand expression by e^{rt}

OK, after all that work, this means we can rewrite the profit maximization expression as:

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$$\underset{T}{MAX}\ P.V.\ \Pi = \frac{PQ(T)}{e^{rT}-1}$$

Taking the derivative of this objective with respect to T gives the following first order condition:

$$FOC: \quad \frac{(e^{rT}-1)(PQ'(T)) - rPQ(T)(e^{rT})}{(e^{rT}-1)^2} = 0$$

This holds if the numerator in the first order condition above is equal to zero, i.e.

$$(e^{rT}-1)(PQ'(T)) = rPQ(T)(e^{rT}) \qquad (3)$$

If we want to show this equilibrium condition to illustrate the decision in a MB=MC format, this can be achieved by dividing equation (3) by e^{rT} and rearrange terms as follows:

$$PQ'(T) = rPQ(T) + PQ'(T)e^{-rT} \qquad (4)$$

Equation (4) tells us that at the optimal rotation length, the gain in waiting an additional unit of time (i.e., the marginal revenue gained by waiting to harvest) is equal to the cost of waiting (namely, (1) the interest payments lost between now and the next time period if trees are harvested in the next time period rather than now, and (2) the present value of the change in revenue of the next rotation due to a change in rotation length).

By substituting the parameters given in the problem, the equilibrium condition can be applied to the problem at hand:

$$10(24T - T^2) = (0.1)(10)(12T^2 - \tfrac{1}{3}T^3) + 10(24T - T^2)e^{-0.1T} \qquad (5)$$

Since you are not asked to solve for T*, it was only necessary to work up to equation (5).

c) *In the case of the repeated rotation, will the optimal rotation length increase or decrease when P increases? Why? What if P remained the same, but r increases. Does T* increase or decrease in this case? Why?*

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[2] The sum of the infinite series : $1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$

If P=$15 instead of $10 as the problem stated, our equilibrium condition (equation (4)), tells us that the optimal rotation length won't change, since P cancels from both sides of this equation. The marginal revenue gained in waiting to harvest increases, but so does the marginal cost of waiting. However, if r increases, then optimal rotation length will decrease, since the interest payments are now higher.

2. <u>*Background:*</u>

NOTE: This was a review question about externality.

Fox Canyon is a fertile agricultural region extending all the way down to the coast, enveloping the bustling metropolis of Oxnard California. It is one of the premier strawberry growing regions of the country, and relies heavily on groundwater from wells for crop production. Because the canyon ends at the seacoast, when a grower pumps water from a well, some sea water bleeds in the well water and makes the well water a little more salty for everyone in the Canyon. This salt hurts plant growth and therefore increases the cost of production for all growers in the area.

The Fox Canyon Groundwater Management Agency (FCGMA) represents all of the growers in the area. Their goal is to manage the water in a way that is best for all the growers.

Suppose the farmers' demand for water is represented by the following inverse demand function $P=P(Q) = 480-0.005Q$, where Q is in acre-feet (an acre-foot is the amount of water it takes to cover one acre of land with one food of water)

The energy cost of pumping water to the surface is $80 per acre-foot (the marginal cost of providing water)

Assume the cost imposed on growers using well water due to the increase in salinity for one acre-foot extracted is $150.

Suppose you are asked to present a report for the FCGMA analyzing a variety of management options.

OPTIONS TO ANALYZE

a. CURRENT SITUATION
Suppose the costs due to increase in salinity are ignored by the FCGMA and no limitations are put on pumping

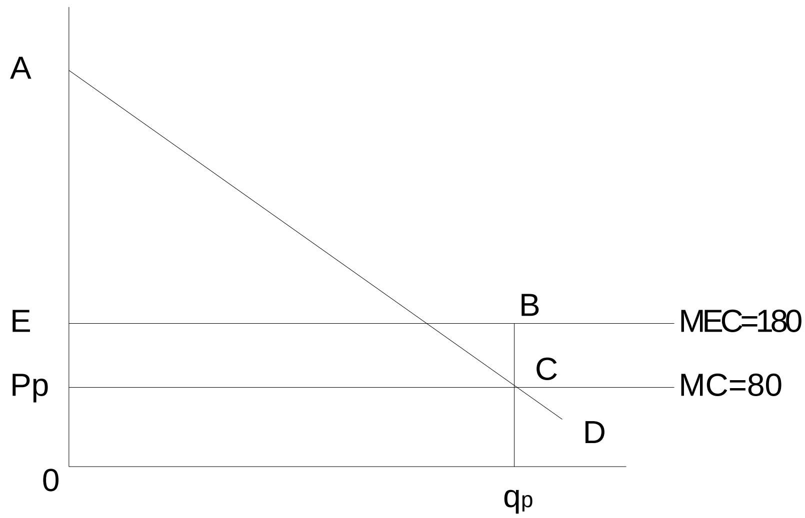
i) What inefficiencies arise from the situation under the current policy? Explain why production is not at the optimal level. Use economic terms and put your answer in a form that a non-economist from FCGMA could understand

The current situation is inefficient because a negative externality is associated with water extraction. The negative externality is in the form of the cost due to increased salinity that a grower imposes on everyone else when that grower extracts well water.

Since growers don't face appropriate costs, they will not correctly account for this damage in their decision-making. They will use too much water compared to the socially optimal amount, in which all costs are correctly incorporated.

The cause of this negative externality is the lack of established and enforceable individual property rights to clean groundwater. Groundwater is currently an open access resource, meaning that anyone can extract as much as they want to. In order for property rights to be effective, the current situation would have to change to make them enforceable using the information gathered on groundwater pumping.

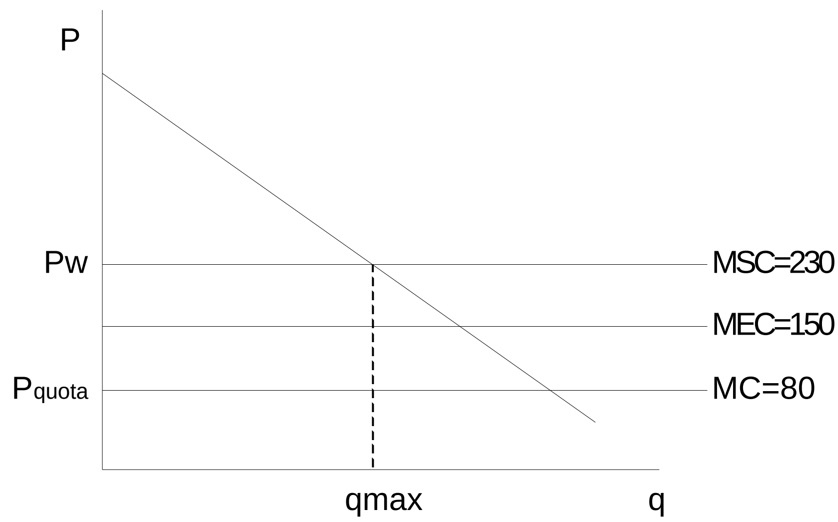
ii) *Sketch a diagram illustrating the quantity of water pumped, consumer surplus, producer surplus, market imperfection and the total cost of that market imperfection.*



Consumer Surplus is area ACPp
Producer Surplus = 0 (in general, the area between price and marginal cost curve)
Cost of externality is represented by area EBq_p0

b. *WATER PUMPING LIMITS*
The primary option that FCGMA is considering is to put limits on the maximum amount of water that growers can pump, Q-max.

i) *Sketch a diagram with the optimal Q-max. Make sure to show what it costs growers to extract water. Is the marginal value of water to each grower the same as the marginal cost of extraction at the allocated quantity? Explain*



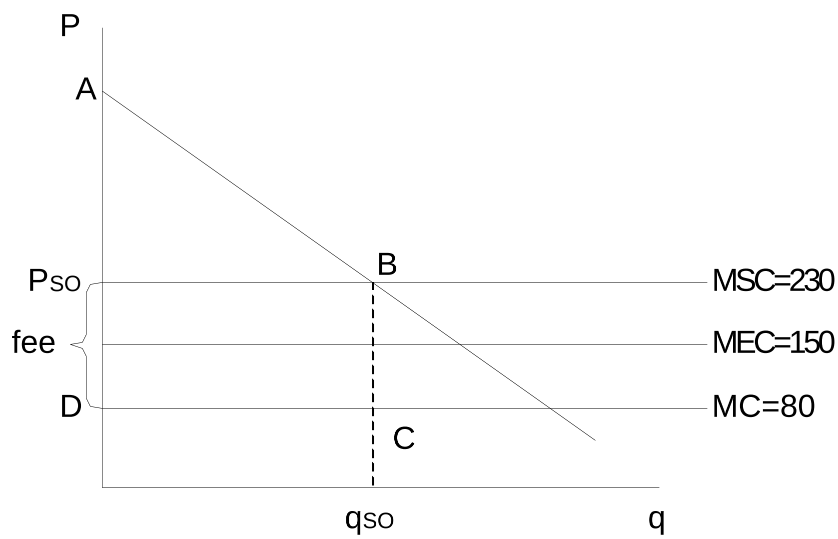
Growers pay P_{quota}, the pumping cost. They would be willing to pay Pw but don't have to.

c. PUMPING FEES
Another option is to charge the growers a fee for each acre-foot pumped.

i) *In words, to what should this fee be set equal?*

The fee should be set to the damage that is imposed on growers and not incorporated by the market. This is the marginal externality cost of pumping well water (MEC) at social optimum.

ii) Sketch a diagram showing the equilibrium with this fee; show CS, PS, and the revenue generated to the FCGMA.



CS corresponds to area ABP_{SO}
PS is zero
Revenue corresponds to area $BCDP_{SO}$

iii) *Calculate the competitive equilibrium quantity of water, marginal extraction cost, and per acre-foot fee using the demand and costs given in the problem description.*

Without fee

$$\underset{q}{MAX} \quad \Pi = P * q - MC * q$$

$$FOC: \frac{d\Pi}{dq} = 0 \ ? \quad P = MC \ ? \quad P = 80$$

$$? \quad 480 - 0.005q = 80$$

$$? \quad q = 80{,}000 \ A.F.$$

Fee Set at MEC, or $150 per AF

$$\underset{q}{MAX} \quad \Pi = pq - (MC + MEC)\,q$$

$$FOC: \frac{d\Pi}{dq} = 0 \ ? \quad P = MC + MEC = MSC \ ? \quad P = 230$$

$$? \quad 480 - 0.005q = 230 \ ? \quad q_{soc}* = 50{,}000 \ A.F.$$