Lecture 2

Elements of optimization theory

2.1 Kuhn-Tucker Conditions

With reference to figure 2.1, consider the problem

$$\max_x f(x)$$

s.t.

$$x \geq 0$$

when $f(\cdot)$ is concave.

In the case of f^1 the optimum occurs at x_1, where $f_x^1(x_1) = 0$. In the case of f^2 the global maximum is not achievable, so that the constrained optimum occurs at $x = 0$ where $f_x^2(0) < 0$.

Therefore, in the most general case, the F.O.C. must be characterized by:

$$f_x(x^*) \leq 0, \qquad x^* f_x(x^*) = 0$$

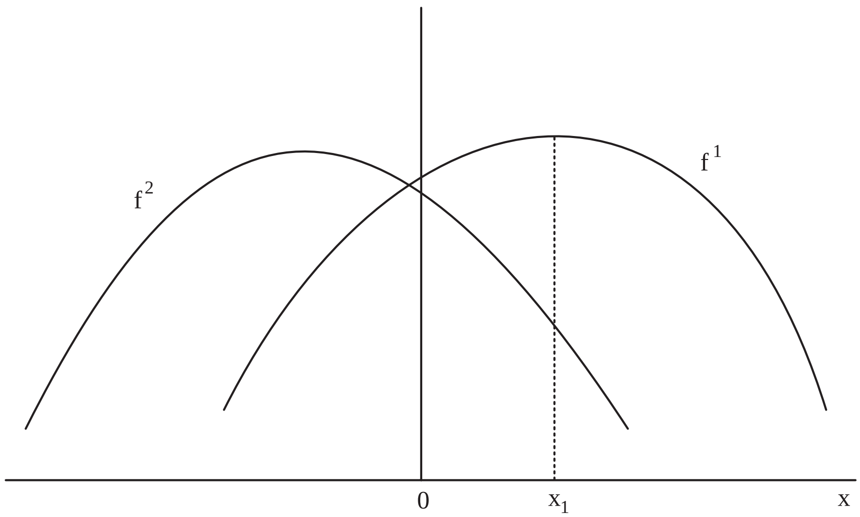
Consider now

$$\max_{\mathbf{x}} f(\mathbf{x}) \tag{2.1}$$
$$\text{s.t.}$$
$$g(\mathbf{x}) \leq \mathbf{b}$$
$$\mathbf{x} \geq \mathbf{0}$$

Let $\mathbf{z} = \mathbf{b} - g(\mathbf{x})$, with $\mathbf{z} \geq \mathbf{0}$, denote a vector of slack variables. The optimization problem becomes

$$\tilde{L} = \max_{\mathbf{x},\mathbf{z},\boldsymbol{\lambda}} f(\mathbf{x}) + \boldsymbol{\lambda}' \left[\mathbf{b} - \mathbf{z} - g(\mathbf{x})\right]$$

Figure 2.1: A constrained maximization problem



s.t. $\mathbf{x} \geq 0$ and $\mathbf{z} \geq 0$.

The first-order conditions are

$$\tilde{L}_x \leq \mathbf{0} \quad \Rightarrow \quad f_x(\mathbf{x}^*) - \boldsymbol{\lambda}^{*\prime} g_x(\mathbf{x}^*) \leq \mathbf{0} \tag{2.2}$$

$$\mathbf{x} \cdot \tilde{L}_x = \mathbf{0} \quad \Rightarrow \mathbf{x}^* \left[f_x(\mathbf{x}^*) - \boldsymbol{\lambda}^{*\prime} g_x(\mathbf{x}^*\right] = \mathbf{0} \tag{2.3}$$

$$\tilde{L}_z \leq \mathbf{0} \quad \Rightarrow \quad -\boldsymbol{\lambda}^* \leq \mathbf{0} \tag{2.4}$$

$$\mathbf{z}^* \cdot \tilde{L}_z = \mathbf{0} \quad \Rightarrow \quad \boldsymbol{\lambda}^{*\prime} \cdot \mathbf{z}^* = \mathbf{0} \tag{2.5}$$

By replacing $\mathbf{z}^*$ with $\mathbf{b} - g(\mathbf{x}^*)$, and defining the lagrangian L as

$$L = f(\mathbf{x}) + \boldsymbol{\lambda}'[\mathbf{b} - g(\mathbf{x})],$$

the optimality condition of (2.2)-(2.5) become

$$L_x = f_x(\mathbf{x}^*) - \boldsymbol{\lambda}^{*\prime} g_x(\mathbf{x}^*) \leq \mathbf{0} \tag{2.6}$$

$$\mathbf{x}^* \cdot L_x = \left[\mathbf{x}^* f_x(\mathbf{x}^*) - \boldsymbol{\lambda}^{*\prime} g_x(\mathbf{x}^*)\right] = \mathbf{0} \tag{2.7}$$

$$L_\lambda = \mathbf{b} - g(\mathbf{x}^*) \geq \mathbf{0} \tag{2.8}$$

$$\boldsymbol{\lambda}^{*\prime} L_\lambda = \boldsymbol{\lambda}^{*\prime} [\mathbf{b} - g(\mathbf{x}^*)] = \mathbf{0} \tag{2.9}$$

$$\mathbf{x} \geq \mathbf{0} \tag{2.10}$$

$$\boldsymbol{\lambda} \geq \mathbf{0}. \tag{2.11}$$

This formulation is valid for multidimensional problems. These conditions are sufficient if $f(\cdot)$ is quasiconcave and $g(\cdot)$ is quasiconvex.

2.2 Linear Programming and constrained optimization

2.2.1 Primal vs. Dual

Consider the following optimization problem:

$$\max_{\mathbf{x}} \mathbf{c}'\mathbf{x} \tag{2.12}$$

s.t.

$$A\mathbf{x} \leq \mathbf{b} \tag{2.13}$$

and

$$\mathbf{x} \geq \mathbf{0}.$$

where $\mathbf{x}$ is a $(n \times 1)$ vector of choice variables, corresponding to the levels of possible production activities; $\mathbf{c}$ is the $(n \times 1)$ vector of net returns of the activities $\mathbf{x}$; A is the $(m \times n)$ matrix of technical coefficients, where the element $\{a_{ij}\}$ indicates the requirement of resource i needed to activate one unit of the process j; and $\mathbf{b}$ is the $(m \times 1)$ vector of resource availability.

In this problem we have n variables and m constraints, with $n > m$.

It is an optimization problem subject to inequality constraints. An optimal solution can be characterized by the necessary *Kuhn-Tucker* conditions.

By substituting the constraint (2.13) in the objective function, write:

$$L = \mathbf{c}'\mathbf{x} + \boldsymbol{\lambda}'[\mathbf{b} - A\mathbf{x}]$$

where $\boldsymbol{\lambda}$ is an $(m \times 1)$ vector of *shadow prices.*

The optimality conditions are given by:

$$L_{\mathbf{x}} = \mathbf{c} - A'\boldsymbol{\lambda} \leq \mathbf{0},$$

$$L_{\boldsymbol{\lambda}} = \mathbf{b} - A\mathbf{x} \leq \mathbf{0}$$

and the following *complementary slackness conditions*:

$$L'_{\mathbf{x}}\mathbf{x} = 0 \quad \Rightarrow [\mathbf{c}' - \boldsymbol{\lambda}'A]\mathbf{x} = 0$$

and

$$\boldsymbol{\lambda}'L_{\boldsymbol{\lambda}} = 0 \quad \Rightarrow \boldsymbol{\lambda}'[\mathbf{b} - A\mathbf{x}] = \mathbf{0}$$

This problem is called the *primal*.

Now consider the following:

$$\min_{\boldsymbol{\lambda}} \boldsymbol{\lambda}'\mathbf{b}$$

subject to:

$$A'\mathbf{x} \geq \mathbf{c}.$$

and:

$$\mathbf{x} \geq 0.$$

This is called the *dual*.

The fundamental result in linear programming is that the primal and the dual have the same first order necessary conditions.

These lead to some of the following results:

1. At the optimal solution, $\mathbf{c}'\mathbf{x}^* = \boldsymbol{\lambda}^{*\prime}\mathbf{b}$ or, in scalar notation,

$$\sum_{i=1}^{n} c_i x_i^* = \sum_{j=1}^{m} \lambda_j^* b_j;$$

2. if $\sum_{i=1}^{n} a_{ij} x_i < b_j$, then $\lambda_j^* = 0$;
3. if $\sum_{j=1}^{m} a_{ij} \lambda_j > c_i$, then $x_i^* = 0$.

2.2.2 The bordered Hessian and its applications

Consider the following optimization problems:

$\max_{\mathbf{x}} f(\mathbf{x})$	$\min_{\mathbf{x}} f(\mathbf{x})$
s.t.	s.t.
$g(\mathbf{x}) = \mathbf{b}$	$g(\mathbf{x}) = \mathbf{b}$

where $\mathbf{x}$ is an $(n \times 1)$ vector of decision variables, $\mathbf{b}$ is an $(m \times 1)$ vector of resource availability, $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$.

Using Lagrange multiplier techniques, these problems become

$\max_{\mathbf{x}} f(\mathbf{x}) + \boldsymbol{\lambda}'[\mathbf{b} - g(\mathbf{x})]$	$\min_{\mathbf{x}} f(\mathbf{x}) + \boldsymbol{\lambda}'[\mathbf{b} - g(\mathbf{x})]$

where $\boldsymbol{\lambda}$ is a $(m \times 1)$ vector of Lagrange multipliers. The first-order conditions for both problems are:

$$\underbrace{\nabla_x f(\mathbf{x})}_{(n\times 1)} - \underbrace{\nabla_x g(\mathbf{x})}_{(n\times m)} \underbrace{\boldsymbol{\lambda}}_{(m\times 1)} = \underbrace{\mathbf{0}}_{(n\times 1)}$$

$$\underbrace{\mathbf{b} - g(\mathbf{x})}_{(m\times 1)} = \underbrace{\mathbf{0}}_{(m\times 1)}$$

The bordered Hessian of these problems is:

$$H = \left[\begin{array}{c|c} \overbrace{\nabla^2_{xx} f(\mathbf{x}) - \nabla_x \left[\nabla_x g(\mathbf{x})\boldsymbol{\lambda}\right]}^{(n\times n)} & \overbrace{-\nabla_x g(\mathbf{x})}^{(n\times m)} \\ \hline \underbrace{-\nabla_x{}' g(\mathbf{x})}_{(m\times n)} & \underbrace{\mathbf{0}}_{(m\times m)} \end{array}\right].$$

The principle minor of order k is a submatrix where the same k rows and columns are deleted. None of the deleted rows and columns can belong to the border.

More formally, let H be the bordered Hessian; $\bar{H}_{ijk,ijk}$ is a submatrix of H without rows ijk and columns ijk. $\bar{H}_{ijk,ijk}$ is a principle minor of rank 3 if i, j, or k is smaller or equal to n; i.e., they are not in the border. The second-order conditions of the problems on page 1 are

maximum	minimum
$\text{sign}[\det\,(H)] = (-1)^n$	$\text{sign}[\det(H)] = (-1)^m$
$\text{sign}[\det(\text{P.M.order } 1)] = (-1)^{n-1}$	$\cdot$
$\ldots$	$\ldots$
$\text{sign}[\det(\text{P.M.order } n-m+1)] = (-1)^{m+1}$	$\text{sign}[\det(\text{P.M.order } n-m+1)] = (-1)^m$

2.2.3 Example of Application

Consider the problem

$$\max_{\mathbf{x}} f(\mathbf{x})$$
$$\text{s.t.}$$
$$g(\mathbf{x}) = \mathbf{b}$$

with $\mathbf{x} \in \mathbb{R}^3$, $\mathbf{b} \in \mathbb{R}^2$, $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^3 \to \mathbb{R}^2$, that is, a problem with three variables and two constraints. The Lagrangian for this problem can be formed as

$$L = \max f(\mathbf{x} + \lambda^1 \left[b_1 - g^1(\mathbf{x})\right] + \lambda^2 \left[b_2 - g^2(\mathbf{x})\right]$$

with first order conditions:

$$\nabla_x f(\mathbf{x}) - \nabla_x g(\mathbf{x})\boldsymbol{\lambda} = \mathbf{0}$$
$$\mathbf{b} - g(\mathbf{x}) = \mathbf{0}$$

or, in scalar notation,

$$
\begin{aligned}
f_{x_1} - g^1_{x_1}\lambda_1 - g^2_{x_1}\lambda_2 &= 0 \\
f_{x_2} - g^1_{x_2}\lambda_1 - g^2_{x_2}\lambda_2 &= 0 \\
f_{x_3} - g^1_{x_3}\lambda_1 - g^2_{x_3}\lambda_2 &= 0 \\
b_1 - g^1(\mathbf{x}) &= 0 \\
b_2 - g^2(\mathbf{x}) &= 0
\end{aligned}
$$

and the bordered Hessian would be:

$$
H = \begin{bmatrix}
(f_{11} - g^1_{11}\lambda_1 - g^2_{11}\lambda_2) & (f_{12} - g^1_{12}\lambda_1 - g^2_{12}\lambda_2) & (f_{13} - g^1_{13}\lambda_1 - g^2_{13}\lambda_2) & (-g^1_1) & (-g^2_1) \\
(f_{21} - g^1_{21}\lambda_1 - g^2_{21}\lambda_2) & (f_{22} - g^1_{22}\lambda_1 - g^2_{22}\lambda_2) & (f_{23} - g^1_{23}\lambda_1 - g^2_{23}\lambda_2) & (-g^1_2) & (-g^2_2) \\
(f_{31} - g^1_{31}\lambda_1 - g^2_{31}\lambda_2) & (f_{32} - g^1_{32}\lambda_1 - g^2_{32}\lambda_2) & (f_{33} - g^1_{33}\lambda_1 - g^2_{33}\lambda_2) & (-g^1_3) & (-g^2_3) \\
(-g^1_1) & (-g^1_2) & (-g^1_3) & 0 & 0 \\
(-g^2_1) & (-g^2_2) & (-g^2_3) & 0 & 0
\end{bmatrix}
$$

Comparative Statics

By totally differenciating the first order conditions, the following comparative statics results can be derived:

$$
H \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \\ d\lambda_1 \\ d\lambda_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} db_1 \\ db_2 \end{bmatrix}
$$

$$
\begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \\ d\lambda_1 \\ d\lambda_2 \end{bmatrix} = H^{-1} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} db_1 \\ db_2 \end{bmatrix}
$$

solving, for example, for dx_3/db_1, one obtains:

$$
\frac{dx_3}{db_1} = \left[H^{-1}_{3\cdot}\right] \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} = -H^{-}1_{35} = -(-1)^{5+3}\frac{\det \bar{H}_{4,3}}{\det H}
$$

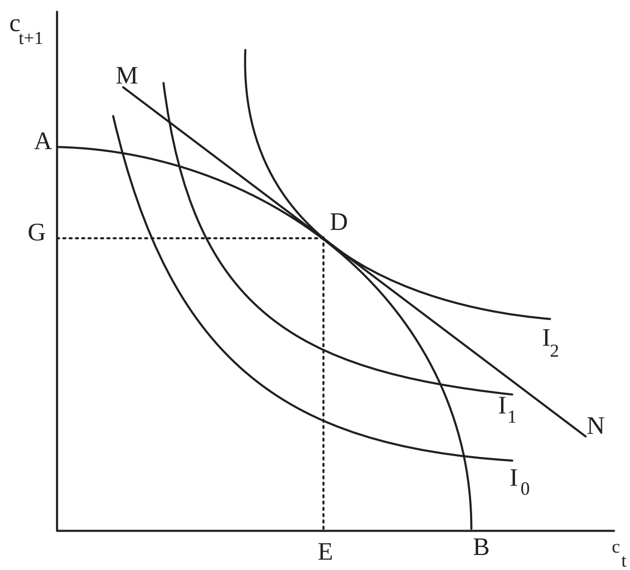
where $\bar{H}_{4,3}$ is a submatrix of H without row 4 (which corresponds to b_1) and column 3 (which corresponds to x_3). The determinant of H is positive, from the second order conditions, but the determinant of $\bar{H}_{4,3}$ is unknown.

2.3 Decision Making Over Time

A meaningful analysis of the agricultural and resource system cannot be done without understanding their evolution over time. One of the most important decisions economists are asked to evaluate are investment decisions, and that entails assessment of dynamic flows of incomes and expenditures. This section will introduce the main elements of a dynamic system, and methodologies for decision making over time including optimal control. We will start, however, discussing the most important concepts in dynamic economic analysis —**discounting** and the **interest rate**.

There is a basic theoretical explanation for determination of the interest rate (sometimes we will use the term discount rate). This is an equilibrium price for a one- period delay in use of one unit of value (a dollar). Figure 2.2 illustrates the determination of the discount rate.

Figure 2.2: Consumption decision over time



The curve AB is a production possibility frontier, and it denotes all the possible tradeoffs between consumption in period t and period $t+1$. Point B corresponds to a situation with maximum consumption in period t and no consumption in period $t+1$. At A all the consumption is delayed to period $t+1$, and the points connecting

A to B correspond to positive consumption in both periods. The figure is drawn under the assumption that delay in consumption will lead to expanded resource availability. At point D, BE resource units are not consumed in period t and lead to the availability of OG units of resources in period $t+1$. The assumed concavity of the tradeoff curve is the reason for this outcome. Each of the curves I_0, I_1, and I_2 is a locus of consumption patterns that result in the same level of utility; therefore, consumers are indifferent to movements between points on the same curve. Higher indifference curves correspond to higher utility levels. The optimal resource allocation is at D where the highest feasible indifference curve is tangent to the production possibility frontier. The slope of the tangency line MN is $\frac{1}{1+r}$, and r is the discount rate.

Under certain assumptions (competition, full information, no externalities), the market gives rise to the socially optimal discount rate. These conditions are not likely to hold in many cases, and market discount rates are likely to be different and, in most cases, higher than the social interest rate. The importance of monetary policy considerations in the determination of interest rates is one reason for the difference between market rates and socially optimal rates. Market rates may be higher for several reasons:

1. **Risk considerations**.– Society has a much better capacity than any individual to carry risks, especially ones that are not correlated. Because society has many members that carry and share uncorrelated risks, the risk cost to each individual approaches zero, and societal choices should be conducted using an interest rate reflecting risk neutrality. Many market rates reflect risk aversion[1].

2. **Externality considerations**.– When individuals decide about future investments, they consider only the benefits to themselves and their direct families in the future. However, their sons or daughters may get married, and other unrelated people may benefit from the activities generated by this investment. However, these benefits are not taken into account by the private parties. Therefore, they tend to underinvest from a social perspective and that can be interpreted as having a private interest rate that is bigger than the social one (note that higher interest rates mean that benefits in the present are weighted more heavily relative to benefits in the future)[2].

The interest rates paid by producers and consumers are adjusted to incorporate a lot of other elements besides society's equilibrium value of time preference. They

[1]This argument was first introduced by the Arrow and Lind paper which is Chapter 11 in Arrow's book.

[2]This argument was advanced by Marglin.

have to incorporate factors that adjust for inflation, transaction costs associated with facilitation of loans, etc.

2.3.1 The Decomposition of Interest Rates

I = Interest rates paid by individuals on organizations which can be decomposed into several elements:

- Real market discount rate: R
- Rate of inflation: IR
- Transaction cost: TC
- Risk factor: SR

$I = R + IR + TC + SR$

Examples:

- Banks pay to Federal Reserve: $R + IR$.
- Best customers of banks (with lowest risk) pay prime interest rate: $I = R + IR + TC_m + SR_m$
  (TC_m and SR_m factors reflect *minimum* transaction and risk cost levels).
- Loans backed by assets generally pay lower interest rates than loans that are not backed by assets.
- Credit ranking and other devices are used by lenders to assess riskiness of loan and to determine the risk factor.
- Lenders assess investment and new projects before financing them. They do it to assess riskiness.

Examples:

1. If the inflation rate is 4%:

   ⇒ Federal reserve loan or borrow to banks at 7%.

   ⇒ Real interest rate is 3%.

   If prime interest rate is 8%:

   ⇒ Risk and transaction cost of banks is 1%.

2. If home mortgage loans are 9%:

   ⇒ Lender receives 7%.

   ⇒ Risk-transaction cost is 2%.

3. If the nominal interest rate is 12%:

   ⇒ Inflation rate is 14%.

   ⇒ Real interest rate is -2%.

When interest rates are higher, individuals are less likely to invest money in projects and instead will deposit their money in banks and buy government bonds.

Methodology to Assess Investments

Step 1. Projection of economic impact: assess cost and benefit over time.

Step 2. Use discount rate to compute net present value or compute internal rates of return.

Investment Criteria

- Net present value:

$$\max \sum_{t=0}^{N} \left(\frac{1}{1+r}\right)^t (B_t - C_t)$$

  or

$$\max \int_0^T e^{-rt}(B_t - C_t)dt.$$

  where B_t are benefits, C_t are costs.

- Internal rate of return —x solves the equation

$$\sum \left(\frac{1}{1+x}\right)^t (B_t - C_t) = 0$$

First Example:

Year	0	1	2
Net benefit	-100	66	60.5
Discounted benefit 10%	-100	$\frac{66}{1.1} = 60$	$\frac{60.5}{1.21} = 50$
NPV	110		

Internal rate of return is solved from:

$$100 = \frac{66}{1+x} + \frac{60.5}{(1+x)^2}$$

$$\Rightarrow 100x^2 + 134x + 26.5 = 0$$

$$\Rightarrow x = \frac{-134 \pm \sqrt{(134)^2 + 400 \cdot 26.5}}{200}$$

$$\Rightarrow x = \frac{-134 + 169}{200} = 0.175$$

Second Example:

Year	0	1	2
Net benefit	-100	70	70

$$100 = \frac{70}{1+x} + \frac{70}{(1+x)^2}$$

$$y = (1+x)$$

$$10y^2 - 7y - 7 = 0$$

$$y = \frac{7 \pm \sqrt{49 + 280}}{20}$$

$$y = \frac{7 \pm 18}{20} = \begin{cases} y_1 = 1.25 \\ y_2 = -0.55 \end{cases} \Rightarrow y = 1.25, x = 0.25$$

Paradox of Internal Rate of Return

Consider a flow of net benefits $\{-a, b, -c\}$. The internal rate of return, z, solves

$$-a + \frac{b}{1+z} - \frac{c}{(1+z)^2} = 0$$

The quadratic equation may have two positive solutions.

Suppose for example that $a = 10$, $b = 30$ and $c = 20$. In such a case:

$$-10(1+z)^2 + 30(1+z) - 20 = 0 \Rightarrow 1 + z = \frac{30 \pm \sqrt{100}}{20} = \frac{30 \pm 10}{20}$$

$$\Rightarrow (1+z) = \begin{cases} 1 \\ 2 \end{cases}$$

What is the true z?

Conclusion: How to analyze investments?

1. Use internal rate of return only in cases when there is one switch. Namely, investment occurs first and returns later.

2. Use net present value in most cases.

2.3.2 Dynamic Systems

A dynamic system can be represented by the following

1. Components:

   stock variable s_t,

   policy variable x_t,

   parameters p_t,

   random noise.

2. Relationships:

   equation of motion, $g_t(s_t, x_t)$

   objective function $f_t(x_t, s_t, p_t)$,

   constraints,

   initial conditions.

3. Formulation of a deterministic control model,

$$\max_{s_t} \sum_{t=0}^{T} \left(\frac{1}{1+r} \right)^t f_t(x_t, s_t, p_t) + V(s_t)$$

   s.t.

$$s_{t+1} - s_t = g_t(x_t, s_t, p_t) \qquad t = 0, \ldots, T$$

   given S_0.

4. Examples:

   - economic growth models,
   - water management problems.

2.3.3 Optimal Control

Optimal control is used to derive optimal policies for a dynamic system. The analytic methodology for deriving the optimal solution to deterministic dynamic problems was developed by Pontrigin and has been applied to a wide range of economic problems. We will first obtain optimality conditions for discrete optimal control problems and then will analyze the necessary optimality conditions of a continuous control problem.

The deterministic problem is

$$\max_{\mathbf{x}_t} \sum_{t=0}^{T} \beta^t f(\mathbf{x}_t, \mathbf{s}_t, \boldsymbol{\theta}_t) + \beta^{T+1} V(\mathbf{s}_{T+1})$$

subject to

$$\mathbf{s}_{t+1} - \mathbf{s}_t = g(\mathbf{s}_t, \mathbf{x}_t), \qquad t = 0, \dots, T$$

where
$\mathbf{x}_t$ = vector of control variables;
$\mathbf{s}_t$ = vector of stock variables;
$\beta = (1 + r)$ is a discounting coefficient, and
$\boldsymbol{\theta}_t$ = a vector of parameters in time t.

For example, $f(\mathbf{x}_t, \mathbf{s}_t, \boldsymbol{\theta}_t)$ can be the revenue from a fishing operation. $\mathbf{s}_t$ may be the stock of fish, $\mathbf{x}_t$ the fishing effort; $f(\cdot)$ may be profit = revenue - cost; and $\boldsymbol{\theta}_t$ is a vector of prices. The equation of motion may denote the change in the fishing population and may combine the effect of fishing (reduces stock) and growth. The function, $V(\mathbf{s}_{T+1})$, is the terminal value of the stock of fish at the end of the planning horizon.

The Langrangian approach may be useful for solving this problem (we omit the θ's for convenience).

$$L = \max_{x_t, s_t, \lambda_t} \sum_{t=0}^{T} \beta^t \quad \underbrace{f(s_t, x_t)}_{\substack{\text{net benefit} \\ \text{period } t}} \quad -\lambda_t[s_{t+1} - s_t - g(s_t, x_t)] + \beta^{T+1} \underbrace{V(s_{T+1})}_{\text{scrap value}}$$

subject to the initial condition $s_0 = \bar{s}_0$.

The first-order conditions are

$$L_{x_t} = \beta^t f_{x_t} + \lambda_t g_{x_t} = 0, \quad t = 0, \dots, T \tag{2.14}$$

$$\dot{s}_t = \frac{\partial H}{\partial \lambda} = g(s_t, x_t) L_{s_t} =$$

$$\beta^t f_{s_t} f_{s_t} + \lambda_t [1 + g_{s_t}] - \lambda_{t-1} = 0, \quad t = 1, \dots, T \tag{2.15}$$

$$L_{s_{T+1}} = \beta^{T+1} V_{s_{T+1}} - \lambda_T = 0 \tag{2.16}$$

$$L_{\lambda_t} = s_{t+1} - s_t - g(s_t, x_t) = 0, \quad t = 0, \dots, T \tag{2.17}$$

given S_0.

The Lagrange coefficient, λ_t, denotes the marginal value of extra units of stock at the end of period t discounted to period 0. It is the shadow price of the equation of motion.

Condition (2.14) suggests that the decision variable, x_t, is set so that its discounted marginal net benefit at period t is equal to its marginal costs in terms of stock growth. We expect $g_{x_t} < 0$.

Conditions (2.15) and (2.16) can be rewritten

$$-(\lambda_t - \lambda_{t-1}) = \beta^t f_{s_t} + \lambda_t g_{s_t}, \qquad t = 1, \dots, T \tag{2.18}$$

$$\lambda_T = \beta^{T+1} V_{s_{T+1}} \tag{2.19}$$

Equations (2.18) and (2.19) provide a set of difference equations establishing the dynamics of the shadow price of the stock. The shadow price of time T is equal to the discounted marginal contribution of the residual stock at time $T+1$. (All shadow prices are discounted to time 0, so we do not have a time difference problem.)

The net shadow price of stock increases as one approaches zero and the increase $(\lambda_{t-1} - \lambda_t)$ reflects the marginal contribution to production (f_{s_t}) and growth (g_{s_t}) of the stock at this extra period. [Note the *earlier* the stock is introduced, the more it can contribute because it lasts longer.]

The difference equations (2.18) and (2.19) reflect the dynamics of the dual variables (shadow price of stock variables), while the difference equation (2.17) with the initial s_0 sets the dynamics of the real stock variables. Therefore, we have to solve for both the dynamics of (shadow) price and quantities in solving optimal control problems.

To understand the problem better, $f(st, xt)$ may be revenues from production, $p_1 f_1(s_t, x_t) - w_t x_t$, when s_t is fish stock and x_t is effort. The stock equation may be

$$s_t + 1 - s_t = g_1(s_t) - f_1(x_t, s_t)$$

where $g_1(s_t)$ denotes stock growth and $f_1(x_t, s_t)$ is catch. The shadow price, λ_t, is the discounted value of fish *in the water.*

2.3.4 Continuous Optimal Control

It is much more convenient and elegant to use a continuous optimal control model for analytic purposes. We will derive the optimality conditions rather heuristically using the approach of Intrilligator. The continuous version of the model presented earlier is:

$$\max_{\{x_t\}} \int_0^T e^{-rt} f(x_t, s_t) dt + e^{-rT} V(s_T) \tag{2.20}$$

subject to

$$\dot{s}_t = g(s_t, x_t) \tag{2.21}$$

One can write a Langrangian function

$$L = \int_0^T \left\{ e^{-rt} f(x_t, s_t) - \lambda_t \left[\dot{s}_t - g(s_t, x_t) \right] \right\} dt + e^{-rT} V(s_T) \tag{2.22}$$

where λ_t is the dynamic shadow price of the equation of motion. It reflects the discounted marginal value of stock added at time t. This Langrangian function can be rewritten in an alternative way. To see it, note

$$L = \max_{x_t, s_t, \lambda_t} \int_0^\infty e^{-rt} f(x_t, s_t) dt - \int_0^T \lambda_t \dot{s}_t dt + \int_0^T \lambda_t g(s_t, x_t) dt + e^{-rT} V(s_T)$$

since,

$$\int_0^T \lambda_t \dot{s}_t dt = [\lambda_T s_T]_0^T - \int_0^T \dot{\lambda}_t s_t dt$$

using integration by parts. The alternative formulation of the Lagrangian is

$$\begin{aligned} L = \max_{x_t, s_t, \lambda_t} \int_0^T & \left[e^{-rt} f(x_t, s_t) + \lambda_t s_t + \lambda_t g(s_t, x_t) \right] dt \\ & + V(s_T) e^{-rt} - \lambda_T s_T + \lambda_0 s_0 \end{aligned} \tag{2.23}$$

s.t. $s_0 = \bar{s}_0$.

The optimality conditions are obtained by differentiating L, using (2.22) or

(2.23) as appropriate.

$$\frac{\partial L}{\partial x_t} = e^{-rt} f_{x_t} + \lambda_t g_{x_t} = 0, \qquad t = 0, \dots, T-1 \tag{2.24}$$

$$\frac{\partial L}{\partial s_t} = e^{-rt} f_{s_t} + \lambda_t + \lambda_t g_{s_t} = 0, \qquad t = 0, \dots, T-1 \tag{2.25}$$

$$\frac{\partial L}{\partial \lambda_t} = s_t - g(s_t, x_t) = 0, \qquad t = 0, \dots, T-1 \tag{2.26}$$

$$\frac{\partial L}{\partial s_T} = \lambda_T - e^{-rT} V(s_T) = 0 \tag{2.27}$$

$$\frac{\partial L}{\partial \lambda_0} = s_0. \tag{2.28}$$

Pontryagin et al. have derived a simple approach to obtain this optimality condition. They define the Hamiltonian function when

$$H(x_t, s_t, \lambda_t) = e^{-rt} f(x_t, s_t) + \lambda_t g(s_t, x_t)$$

and proved that at each t the optimality conditions are

$$\frac{\partial H}{\partial x_t} = e^{-rt} f_{x_t} + \lambda_t g_{x_t} = 0, \tag{2.29}$$

$$-\dot{\lambda}_t = \frac{\partial H}{\partial \lambda_t} = e^{-rt} f_{s_t} + \lambda_t g_{s_t}, \tag{2.30}$$

$$\dot{s}_t = \frac{\partial H}{\partial \lambda_t} = g(s_t, \lambda_t) \tag{2.31}$$

given S_0 and $\lambda_T = e^{-rt} V(s_t)$.

These sets of conditions allow one to solve a dynamic optimal control problem as a succession of static choice problems corrected by the dynamics of stock variables and their shadow prices. The dynamics of stock is given by (2.30), and $s_0 = \bar{s}_0$ and the shadow prices are presented by (2.31) and $\lambda_T = e^{-rt} V(s_t)$.

2.3.5 Optimal Control Techniques and Applications

In the previous section, we presented the optimality conditions using shadow prices discounted to period 0. Many times we are interested in temporary prices and values, rather than discounted ones. To obtain such prices, we will derive first temporary optimality conditions. Consider the problem discussed earlier

$$\max_{x_t, s_t} \int_0^T e^{-rt} f(x_t, s_t) dt + e^{-rT} V(s_T). \tag{2.32}$$

The (discounted) Hamiltonian of this problem is

$$H^D = e^{-rt} f(x_t, s_t) + \lambda_t^D g(s_t, x_t) \tag{2.33}$$

where λ_t^D is the shadow price of the equation of motion (or the shadow price of stock in time t) in values *discounted* to time 0). The optimality conditions are

$$\frac{\partial H^D}{\partial x_t} = e^{-rt} f_{x_t} + \lambda_t^D g_{x_t} = 0, \tag{2.34}$$

$$-\dot{\lambda}_t^D = \frac{\partial H^D}{\partial s_t} = e^{-rt} f_{s_t} + \lambda_t^D g_{s_t}, \tag{2.35}$$

$$\dot{s}_t = \frac{\partial H^D}{\partial \lambda_t^D} = g(s_t, x_t), \text{ and} \tag{2.36}$$

$$e^{-rt} V_{s_T} = \lambda_T^D \tag{2.37}$$

given S_0.

Let $H_t = e^{rt} H^D$ denote the temporal Hamiltonian and $\lambda_t = e^{rt}\lambda_t^D$ denote the temporal shadow price of the stock. The alternative presentation of the optimality condition is:

$$\frac{\partial H}{\partial x_t} = f_{x_t} + \lambda_t g_{x_t} \quad = \quad 0, \tag{2.38}$$

$$\dot{\lambda}_t + r\lambda_t = \frac{\partial H_t}{\partial s_t} \quad = \quad f_{s_t} + \lambda_t g_{s_t}, \tag{2.39}$$

$$\dot{s}_t \quad = \quad \frac{\partial H_t}{\partial \lambda_t}, \text{ and} \tag{2.40}$$

$$\lambda_t \quad = \quad V_{s_t}(s_t) \tag{2.41}$$

given S_0.

To see that conditions (2.29)–(2.31) and (2.38)–(2.41) are consistent, note that

$$-\dot{\lambda}_t = -\frac{\partial}{\partial t}\left[-r\lambda_t + e^{rt}\dot{\lambda}_t^D\right] = \left[-r\lambda_t + e^{rt}\frac{\partial H^D}{\partial s_t}\right] = \left[-r\lambda_t + \frac{\partial H_t}{\partial s_t}\right].$$

2.3.6 The Simple Model of Economic Growth

The first and perhaps the most important early applications of optimal control in economics pertain to the study of economic growth. These are macro models, but the analysis can apply to micro problems. The economic growth model presented here considers the case when one good is both consumed and invested. Suppose the production function is

$$y_t = f(k_t)$$

where y_t is output and k_t is capital stock. Let c_t be consumption in time t and investment is $y_t - c_t = f(k_t) - c_t$.

Utility is derived with consumption $u(c_t)$ in utility function $u_c > 0$, $u_{cc} \leq 0$. Thus, the objective is

$$\max_{c_t} \int_0^\infty e^{-rt} u(c_t) dt$$

and the equation of motion is

$$\dot{k}_t = f(k_t) - c_t - \gamma k_t$$

Capital increases at the investment rate, but may decline because of depreciation when γ is the depreciation coefficient. The initial capital stock is k_0. The temporary Hamiltonian is

$$H = u(c_t) + \lambda_t \left[f(k_t) - c_t - \gamma k_t \right]$$

and the optimality conditions given by

$$\frac{\partial H}{\partial c_t} = u_c - \lambda_t = 0, \tag{2.42}$$

$$\dot{\lambda}_t + r\lambda_t = \frac{\partial H}{\partial k_t} = \lambda_t f_k - \lambda_t \gamma, \tag{2.43}$$

$$\dot{k}_t = \frac{\partial H}{\partial \lambda_t} = f(k_t) - c_t - \gamma k_t \tag{2.44}$$

First, optimality condition (...) equates the shadow price of the stock to marginal utility of consumption; this is reasonable since a unit of the good is either traded or invested and marginal benefit from both activities has to be equal at the optimal solution. The rate of change in the nominal shadow price is affected by three elements:

1. Discounting (which has the effect of increasing nominal prices over time).
2. Depreciation (which has a similar effect).
3. Marginal productivity of capital (which operates by reducing nominal prices over time). The higher f_k is, the more capital will be available in the future and the less valuable it is.

2.3.7 The Dynamics of Consumption

Total differentiation of (...) yields $u_{cc}\dot{c}_t\dot{\lambda}_t = 0$. From (...), that becomes

$$u_{cc}\dot{c}_t + \lambda_t \left[f_k - r - \gamma \right] = 0.$$

Since $u_c = \lambda_t$, the expression leads to a condition defining the rate of changes in consumption over time.

$$\frac{\dot{c}_t}{c_t} = -\frac{u_c}{u_{cc}c_t}\left[f_k - r - \gamma\right]. \tag{2.45}$$

Let us define $\eta(c_t) = -\frac{u_c}{u_{cc}c_t}$. It can be interpreted as the elasticity of demand for the good. To see this point, suppose we have a utility-maximizing consumer who derives additive utility from c and expenditure. The optimal consumption choice of this individual is

$$\max_c u(c) + I - pc$$

where I is income and p is the commodity price. Optimality condition is $u_c - p = 0$. Total differentiation of this condition yields

$$u_{cc}dc - dp = 0 \Rightarrow \frac{dc}{dp} = \frac{1}{u_{cc}} < 0.$$

Therefore,

$$-\frac{dc}{dp}\frac{p}{c_t} = -\frac{u_c}{u_{cc}c_t} = \eta(c_t).$$

Using this definition, note that

$$\frac{\dot{c}_t}{c_t} = \eta(c_t)[f_k - r - \gamma].$$

If $f_k - r - \gamma > 0$, the productivity of capital is substantial and prices decline over time. Equation (...) suggests that consumption increases and the increase in consumption is inversely related to the demand elasticity.

2.3.8 Steady State

At steady state, both state and co-state (shadow price of stocks) variables do not change over time. Economists are interested in knowing if steady-state situations exist, what will be the dynamic path which leads to them, and whether they are stable. They are stable when the system returns to steady state in spite of some random shocks that move it away.

Economists are enamored with steady states because they are the dynamic equivalent of long-run equilibria. They represent the outcome for which the system converges.

In the growth theory model, steady state occurs when

$$\dot{k} = f(k) - c - \gamma k = 0, \tag{2.46}$$
$$\dot{c} = \dot{\lambda} = f_k(k) - r - \gamma = 0 \tag{2.47}$$

Figure 2.3: A phase diagram

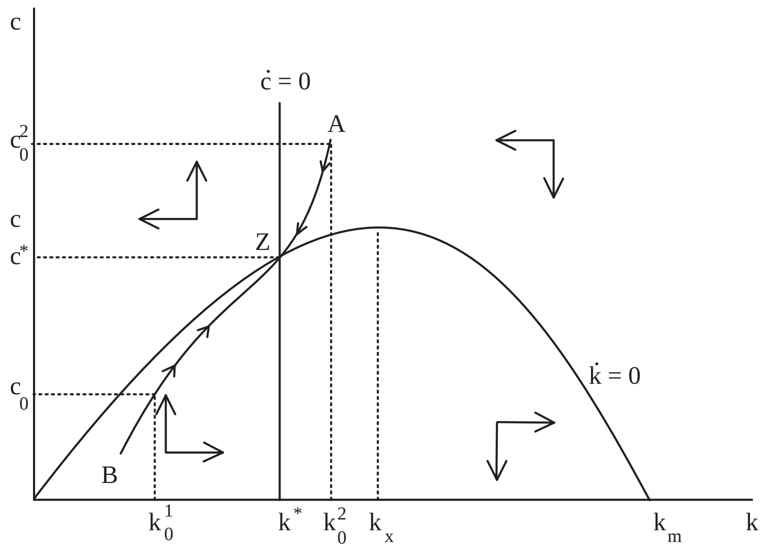


Figure 2.3 presents a phase diagram. The loci of all the k and c combinations that lead to $\dot{k} = 0$ and $\dot{c} = 0$ are depicted as well as their intersection(s) which are the steady states of the system.

The locus of all the points with $\dot{c} = 0$ is at $k = k^*$. The curve $c = f(k) - \gamma k$ is the locus of all the points with $\dot{k} = 0$. It intersects $c = 0$ at $k = 0$ and $k = k_m$. The steady state of the system occurs in our case at Z with $k = k^*$ and $c = c^*$.

To study the dynamic properties of the system, note that the diagram indicates that consumption decreases when $k > k^*$ and consumption increases when $k < k^*$. Similarly, $\dot{k}$ declines above the curve with $\dot{k} = 0$ (when $f(k) - \gamma k - c < 0$) and increases below it.

The curve AZB denotes optimal consumption capital path. If initial capital is k_0^1, initial consumption should be c_0^1 and movement along BZ will lead to equilibrium. If initial capital is k_0^2, initial consumption should be c_0^2 and, for a period before steady state, *consumption* will be greater than *production*.

The analysis here demonstrates the importance of time preference. If $r = 0$, optimal solution will be at k_x. But with $r > 0$, even if initial stock is k_x, there will be periods of excess consumption along the line AZ until the steady-state capital at $k = k^*$ is attained.