Intertemporal Risk Attitude

Lecture 5

▷ Lectures 1-4 familiarized in a simplified 2 period setting with
  ▷ disentangling risk aversion and int. substitutability
  ▷ the concept of intertemporal risk aversion

▷ Lecture 5 introduces
  ▷ a general model
  ▷ a general characterization of intertemporal risk aversion

▷ Lectures 6-8 will discuss
  ▷ axiomatic simplifications of the model
  ▷ a preference for the timing of resolution resolution
  ▷ discounting
Setup - The Choice Space

Time: discrete with arbitrary finite planning horizon $T$

- $X$: (connected compact metric) space of goods
- $\Delta(\cdot)$: Set of Borel \textit{probability} measures on space ‘·’ \textit{(Prohorov metric)}
- $\tilde{X}_T = X$: Degenerate choices in period $T$
- $P_T = \Delta(\tilde{X}_T) = \Delta(X)$: Uncertain choices in period $T$

Recursively define

- $\tilde{X}_{t-1} = X \times \Delta(\tilde{X}_t)$ for all $t \in \{2, \ldots, T\}$ \textit{(product metric)}
- $\tilde{x}_t \in \tilde{X}_t$: degenerate period $t$ choice objects
- $P_t = \Delta(\tilde{X}_t)$ for all $t \in \{1, \ldots, T\}$
- $p_t \in P_t$: general uncertain period $t$ choice objects (lotteries)
Setup - The Choice Space

Certain Choices:

- $X^t = X^{T-t+1}$ denotes the $(T-t+1)$-fold Cartesian product
- $\exists x \in X^t, x = (x_t, x_{t+1}, \ldots, x_T) = (x_t, x_{t+1}, \ldots, x_T)$: certain consumption path from period $t$ to $T$
- Note: $X^t \subset \tilde{X}_t \subset P_t$

Notation: Given $x \in X^t$ define

- $(x_{-1}, x) = (x_t, \ldots, x_{T-1}, x, x_{T+1}, \ldots, x_T) \in X^t$: consumption path coinciding with $x$ in all but the $i^{th}$ period, in which it yields outcome $x$.

Preferences:

- $\succeq_t$: Preference relation in period $t$ on choice space $P_t$
Axioms:

**A1-A3** (VNM axioms) As before, now for $\succeq_t$ on $P_t$ with $t \in \{1, \ldots, T\}$

**A4** (certainty separability)
For all $x, x' \in X^1$, $x, x' \in X$ and $\tau \in \{1, \ldots, T\}$ it holds that

i) $(x_{-\tau}, x) \succeq_1 (x'_{-\tau}, x) \iff (x_{-\tau}, x') \succeq_1 (x'_{-\tau}, x')$

ii) If $T = 2$ additionally with $x_t, x'_t, x''_t \in X$, $t \in \{1, 2\}$:

\[
(x'_1, x_2) \sim_1 (x_1, x''_2) \land (x''_1, x_2) \sim_1 (x_1, x'_2) \Rightarrow (x'_1, x'_2) \sim_1 (x''_1, x''_2)
\]

**A5** (time consistency)
For all $t \in \{1, \ldots, T - 1\}$, $x_t \in X$ and $p_{t+1}, p'_{t+1} \in P_{t+1}$:

\[
(x_t, p_{t+1}) \succeq_t (x_t, p'_{t+1}) \iff p_{t+1} \succeq_{t+1} p'_{t+1}.
\]

**A0** (non-degeneracy) For all $t \in \{1, \ldots, T\}$:

$\exists x \in X^1$ and $x \in X$ such that $(x_{-t}, x) \not\sim_1 x$. 

Given a preference relation $\succeq_t$ on $P_t$ define:

- Induced preferences $\succeq^*_t$ on $X$ by defining for all $x, x' \in X$:
  $$x \succeq^*_t x' \iff (x, x_{t+1}, \ldots, x_T) \succeq_t (x', x_{t+1}, \ldots, x_T) \forall x_{t+1}, \ldots, x_T \in X.$$  

Because of certainty separability $\succeq^*_t x'$ is a complete order on $X$.

- The set of Bernoulli utility functions in period $t$:
  $$B_{\succeq_t} = \{u_t \in C^0(X) : x \succeq^*_t x' \iff u_t(x) \geq u_t(x') \forall x, x' \in X\}$$

Express preference over certain consumption in period $t$. 

Intertemporal Aggregation Rule, define

\[ g = (g_t)_{t\in\{1,\ldots,T\}}: \text{sequence of functions with } g_t \in \mathcal{C}^0(U_t) \]

Each \( g_t : U_t \rightarrow \mathbb{R} \) weighs utility levels in \( U_t \subset \mathbb{R} \)

\[ [G_t, \overline{G}_t] = \text{range}(g_t) \text{ and } \Delta G_t = \overline{G}_t - G_t \]

An intertemporal aggregation rule is a functional

\[ \mathcal{N}_t^g : U_t \times U_{t+1} \rightarrow \mathbb{R} \text{ with } \allowbreak \mathcal{N}_t^g(\cdot, \cdot) = g_t^{-1} [\theta_t g_t(\cdot) + \theta_t \theta_{t+1}^{-1} g_{t+1}(\cdot) + \theta_t \theta_{t+1}^{-1} \vartheta_t] \]

where the normalization constants are defined as

\[ \theta_t = \frac{\Delta G_t}{\sum_{\tau=t}^T \Delta G_{\tau}} \text{ and } \vartheta_t = \frac{\overline{G}_{t+1} G_t - G_{t+1} \overline{G}_t}{\Delta G_t} \]

Remark: \( G_t = 0 \Rightarrow \vartheta_t = 0 \)

In stationary setting \( \theta_t \) relates to discount factor
Let \( \succeq \equiv (\succeq_t)_{t \in \{1, \ldots, T\}} \) on \((P_t)_{t \in \{1, \ldots, T\}}\) satisfy axiom A0 and let \( u \equiv (u_t)_{t \in \{1, \ldots, T\}} \) satisfy \( u_t \in B_{\succeq_t} \). (given Bernoulli utility!)

**Theorem 1:**

The sequence of preference relations \((\succeq_t)_{t \in \{1, \ldots, T\}}\) satisfies A1-A5, if and only if, there exist strictly increasing and continuous functions \(f_t : U_t \rightarrow \mathbb{R} \) and \(g_t : U_t \rightarrow \mathbb{R}\) for \(t \in \{1, \ldots, T\}\) such that with defining recursively the aggregate welfare functions

\[
\tilde{u}_t : \tilde{X}_t \rightarrow \mathbb{R} \text{ by } \tilde{u}_T(x_T) = u_T(x_T) \text{ and for } t \in \{1, \ldots, T-1\}
\]

\[
\tilde{u}_t(x_t, p_{t+1}) = \mathcal{N}_t^g \left( u_t(x_t) , \mathcal{M}_t^{f_{t+1}}(p_{t+1}, \tilde{u}_{t+1}) \right)
\]

it holds for all \(t \in \{1, \ldots, T\}\) that

\[
p_t \succeq_t p'_t \iff \mathcal{M}_t^{f_t}(p_t, \tilde{u}_t) \geq \mathcal{M}_t^{f_t}(p'_t, \tilde{u}_t) \quad \forall p_t, p'_t \in P_t.
\]
Use freedom to choose Bernoulli utility to simplify the representation
\[ \tilde{u}_t(x_t, p_{t+1}) = \mathcal{N}^g_t \left( u_t(x_t), \mathcal{M}^{f_{t+1}}(p_{t+1}, \tilde{u}_{t+1}) \right) : \]

▷ **Kreps-Porteus-form** \((f_t = \text{id})\) (von Neumann-Morgenstern \(u_t\)):
\[ \tilde{u}_t(x_t, p_{t+1}) = \mathcal{N}^g_t \left( u^{\text{NM}}_t(x_t), E_{p_{t+1}} \tilde{u}_{t+1} \right) \]

▷ **Certainty-additive-form** \((g_t = \text{id})\) (certainty additive \(u_t\)):
Normalizing Bernoulli utility to range \([0, \overline{U}_t]\) and absorbing \(\theta_t\) into \(f_t\)
\[ \tilde{u}_t(x_t, p_{t+1}) = u^{ca}_t(x_t) + \mathcal{M}^{f_{t+1}}(p_{t+1}, \tilde{u}_{t+1}) \]

And only in a one commodity world:

▷ **Epstein-Zin-form** \((u_t = \text{id})\):
\[ \tilde{u}_t(x_t, p_{t+1}) = \mathcal{N}^g_t \left( x_t, \mathcal{M}^{f_{t+1}}(p_{t+1}, \tilde{u}_{t+1}) \right) \]
Recall: Risk Aversion & Intertemporal Substitutability

Analysis of Epstein-Zin-form

▷ "\( \mathcal{M}^{f_t, f_t} \)" : describe atemporal (Arrow-Pratt) risk aversion
▷ "\( \mathcal{N}^{g_t, g_t} \)" : describe intertemporal substitutability

BUT: \( f_t \) and \( g_t \) depend on

▷ good under observation
▷ measure scale of good
▷ level of other consumption

⇔ the choice of Bernoulli utility functions in Theorem 1

HOWEVER: The functions \( f_t \circ g_t^{-1} \) are invariant under different choices of Bernoulli utility, and uniquely determined by the preferences \( \succeq_t \) ...

... up to affine indeterminacies of \( f_t \) and \( g_t \).
Intertemporal Risk Aversion - Axiom

For IRA Axiom in general setting:

Define for $x, x' \in X^t$ consumption paths $x^{\text{high}}(x, x'), x^{\text{low}}(x, x') \in X^t$,

$$
(x^{\text{high}}(x, x'))_\tau = \begin{cases} 
 x'_\tau & \text{if } x'_\tau \succeq^*_\tau x_\tau \\
 x_\tau & \text{if } x_\tau \preceq^*_\tau x'_\tau
\end{cases},
$$

$$
(x^{\text{low}}(x, x'))_\tau = \begin{cases} 
 x'_\tau & \text{if } x'_\tau \succeq^*_\tau x'_\tau \\
 x_\tau & \text{if } x'_\tau \succeq^*_\tau x_\tau
\end{cases}
$$

(1)

for $\tau \in \{t, \ldots, T\}$.

$\triangleright$ $x^{\text{high}}(x, x')$: collects better outcomes of every period

$\triangleright$ $x^{\text{low}}(x, x')$: collects inferior outcomes of every period
A decision maker is called \textit{strictly intertemporal risk averse} in period $t < T$ iff

for all $x, x' \in X^t$ satisfying

$$x \sim_t x' \land \exists \tau \in \{t, \ldots, T\} \text{ s.th. } x_\tau \not\succ^*_\tau x'_\tau$$

it holds that

$$x \succ^1\frac{1}{2} x^{\text{high}}(x, x') \succ^1\frac{1}{2} x^{\text{low}}(x, x')$$
Theorem 2:
In the representation of theorem 1, a decision maker is strictly intertemporal risk averse in period \( t < T \), if and only if,
\[ f_t \circ g_t^{-1} \] is strictly concave.

Define the measures of relative intertemporal risk aversion as:
\[
RIRA_t(z) = - \frac{f_t \circ g_t^{-1}''(z)}{f_t \circ g_t^{-1}'(z)} \cdot z,
\]
and the measures of absolute intertemporal risk aversion as:
\[
AIRA_t(z) = - \frac{f_t \circ g_t^{-1}''(z)}{f_t \circ g_t^{-1}'(z)}.
\]

For a particular point in consumption space define
\[
AIRA_t[\tilde{x}_t] = AIRA_t(z) \big|_{z = g_t \circ \tilde{u}_t(\tilde{x}_t)}.
\]
Let the preferences $\succeq = (\succeq_t)_{t \in \{1, \ldots, T\}}$ be represented in the sense of theorem 1 with twice differentiable functions $f_t \circ g_t^{-1}$.

**Theorem 2:**

a) For all $t \in \{1, \ldots, T\}$ choose $\bar{x}_t \in X$ and fix $g_t \circ u_t(\bar{x}_t) = 0$.

Then, the risk measures $\text{RIRA}_t$ are determined uniquely by the preferences $\succeq$ and the point $\tilde{x}_t$ in consumption space. They are independent of the choice of the Bernoulli utility functions.

b) For some arbitrary period $t^* \in \{1, \ldots, T\}$ choose two outcomes $\hat{x}_{t^*}, \check{x}_{t^*} \in X$ with $\hat{x}_{t^*} \succ^* \check{x}_{t^*}$, and fix $g_{t^*} \circ u_{t^*}(\hat{x}_{t^*}) - g_{t^*} \circ u_{t^*}(\check{x}_{t^*}) = 1$.

Then, the risk measures $\text{AIRA}_t$ are determined uniquely by the preferences $\succeq$ and the point $\tilde{x}_t$ in consumption space. They are independent of the choice of the Bernoulli utility functions.
Interpretation:

- The risk measures $RIRA_t$ and $AIRA_t$ depend on a welfare measure scale, where welfare is determined by certain intertemporal trade-offs.

- The intertemporal trade-off welfare scale is only determined up to affine transformations.

Thus, that the measures (recall discussion on measure scale dependence!)

- the measures $RIRA_t$ are uniquely defined if the zero welfare level is fixed
- the measures $AIRA_t$ are uniquely defined if the unit of welfare is fixed

This intertemporal trade-off welfare measure scale for IRA is given by

- $g_t \circ u_t(X) = u_t^{ca}(X)$. 
The certainty additive form is particular convenient for interpretation:

- \( u^c_t \) measures welfare in period \( t \) (determined up to affine trafos)
- \( f_t \) measures intertemporal risk aversion in period \( t \) as risk aversion with respect to welfare gains and losses
- Fixing \( u^c_t(x^0_t) = 0 \) and \( u^c_t(x^1_{t*}) = 1 \) fixes measure scale uniquely

Summarizing functional characterization of IRA:

- Intertemporal Risk Aversion is measured with respect to a measure scale defined up to affine transformations by preferences.
- Therefore: The measures \( \text{RIRA}_t \) and \( \text{AIRA}_t \) are independent of the measure scale for goods, and the good under observation.
- However: For a numerical measure of Intertemporal Risk Aversion unit and/or zero level of the measure scale have to be fixed.
Good and measure scale dependence of risk measures

Space of goods

Coordinate space $\mathbb{R}^n$

coconut quality

litchi quantity

Is there a risk attitude measure that is independent of coordinate system $\Phi$ and the particular good?

I.e. that describes attitude with respect to risk rather than with respect to coconuts, litchis or money?