# Generalized Cost-Benefit-Analysis and Social Discounting with Intertemporally Dependent Preferences

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**Abstract:** We derive a general framework for cost-benefit analysis and social discounting in a setting with intertemporally dependent preferences. Here, the marginal contribution of an additional unit of consumption in some period depends on what is consumed in the other periods. We use a simple model of history dependent preferences to analyze how habit formation affects the social rate of discount. Getting used to good life before a potential decline in growth as well as getting used to bad life during a decline in growth both affect social discount rates, not only during the time of the growth rate change but also before and after.

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# 1 Introduction

We analyze cost-benefit analysis and social discounting in a preference setting with intertemporally dependent preferences. Here the marginal contribution of an additional unit of consumption in some period depends on what is consumed in other periods. Intertemporal preference dependence is integrated into a cost-benefit-framework by using a functional derivative in order to determine the welfare contribution of consumption changes. We use a simple model of history dependent preferences to analyze how habit formation affects the social rate of discount. In a steady state, the discount rates gains an additional term that changes the discount rate depending on the degree of intertemporal sustitutability. We identify two effects that change the evaluation already ahead of times applying the model to a stylized growth scenario with an anticipated change of the growth rate. First, getting used to good life makes the evaluation more sensitive to a period of declining growth. Second, getting used to a relatively lower consumption level in a period of declining growth increases the valuation of an increasing consumption level after the decline.

In section 2 we derive a general approach for cost benefit evaluation in the context of intertemporally dependent preferences. In section 3 we apply the framework to a model of habit formation. In section 4 a special case of the habit formation model is used to analyze the discount rate for intertemporally dependent preferences. This section is currently still incomplete. Appendix A contains calculations, while appendix B shows how a special case of the habit formation model can be translated into the standard model of Hamiltonian optimal control with a 'habit stock'.

# 2 Cost-Benefit-Analysis with Intertemporally Dependent Preferences

## 2.1 Derivation of the General Cost-Benefit-Functional

In this section we evaluate the marginal contribution of an additional unit of consumption in some period to overall welfare. Because we allow for intertemporal dependence of preferences, the marginal contribution of such a consumption unit cannot be determined simply by analyzing its welfare contribution to (instantaneous) utility in that period. Instead we have to account for the welfare changes that such an additional unit of consumption causes in all periods. We achieve this task by replacing the standard derivative with a functional derivative.

We start with a general evaluation functional expressing welfare as a function of the consumption *path*. This generality comes at no additional cost and fleshes out the structure of our approach. Later sections will add more structure to the model. Let $J[c(\cdot)]$ be an evaluation functional for consumption path $c(\cdot)$. For two functions $f, g : \mathbb{R} \to \mathbb{R}$ define $< f, g >\equiv \int_{-\infty}^{\infty} f(t)g(t)\,dt$ (inner product). The Fréchet or functional derivative $\widehat{J[c]}$ of the evaluation functional $J$ at consumption path $c$ is the linear operator characterizing the change of $J[c]$ with respect to some perturbation of the consumption path $\tilde{c}$. Being a linear operator $\widehat{J[c]}$ can be represented by an integral, where the corresponding density function $\widetilde{J[c]}$ is defined by the relation:

$$< \widetilde{J[c]}, \tilde{c} >= \frac{d}{d\epsilon} J[c + \epsilon\,\tilde{c}]\bigg|_{\epsilon=0} \qquad \forall\, \tilde{c} \in C^{\infty}(\mathbb{R})\,.$$

That is, for small perturbations ($\epsilon$ small), the density $\widetilde{J[c]}$ approximates the change of the evaluation functional $J$ with respect to a perturbation $\tilde{c}$ in the same way as the standard derivative $u'(\cdot)$ of a function $u : \mathbb{R} \to \mathbb{R}$ approximates a change in its argument. In both cases we get an approximation of the change by taking the inner product of the density $\widetilde{J[c]}$ respectively derivative $u'(\cdot)$ with the change of the argument.

Let $\delta_t$ denote the delta distribution (not the Dirac Delta!). It is characterized by the relation $< \tilde{c}, \delta_t >= \tilde{c}(t)\ \forall\, \tilde{c} \in C^{\infty}$ and corresponds to a 'density' that concentrates full weight on point of time $t$. Note that $\widetilde{J[c]}(t) = \widehat{J[c]}(\delta_t)$. The welfare change caused by a small deviation $\tilde{c}(t)$ from an overall consumption scenario $c(t)$ can be written as

$$\begin{aligned}\widehat{J[c]}(\tilde{c}) &= \int_{-\infty}^{\infty} \widehat{J[c]}(\delta_t)\,\tilde{c}(t)\,dt \\ &= \int_{-\infty}^{\infty} \widetilde{J[c]}(t)\,\tilde{c}(t)\,dt\,. \qquad (1)\end{aligned}$$

Equation (1) specifies a cost benefit evaluation in continuous time for general evaluation functionals. The function $\widetilde{J[c]}$ depicts the consumption discount factor in welfare units.

The corresponding discount rate calculates to

$$\gamma(t) = -\frac{\frac{d}{dt}\widetilde{J[c]}(t)}{\widetilde{J[c]}(t)} \,.$$

The standard model without intertemporal dependence of preferences assumes that a change of consumption in some period only affects welfare in that period. More precisely it assumes that the evaluation functional $J$ can be represented by a density function that only depends on consumption at time $t$, i.e. $J[c] = \int_{-\infty}^{\infty} U(c(t), t)\, dt$. Then $\widetilde{J[c]}(t) = \frac{\partial U(t)}{\partial c_t}$ and equation (1) simplifies to

$$\widehat{J[c]}(\tilde{c}) = \int_{-\infty}^{\infty} \frac{\partial U(t)}{\partial c_t}\, \tilde{c}(t)\, dt \,, \tag{2}$$

which is the standard evaluation functional underlying cost benefit analysis.

## 2.2 A Two Period Illustration

We illustrate the general cost benefit functional for the case where a project only changes consumption at two points of time, i.e. $\tilde{c} = \delta_{t_1}\tilde{c}(t_1) - \delta_{t_2}\tilde{c}(t_2)$. In order to derive the relative discount factor between consumption in period $t_1$ and period $t_2$ we assume that the simultaneous change in both periods leaves the overall evaluation unchanged:

$$\begin{aligned} & \widehat{J[c]}(\delta_{t_1}\tilde{c}(t_1)) && \stackrel{!}{=} \widehat{J[c]}(\delta_{t_2}\tilde{c}(t_2)) \\ \Leftrightarrow\ & \int_{-\infty}^{\infty} \widetilde{J[c]}(t)\, \delta_{t_1}\, \tilde{c}(t_1)\, dt && = \int_{-\infty}^{\infty} \widetilde{J[c]}(t)\, \delta_{t_2}\, \tilde{c}(t_2)\, dt \\ \Leftrightarrow\ & \frac{\tilde{c}(t_1)}{\tilde{c}(t_2)} && = \frac{\widetilde{J[c]}(t_2)}{\widetilde{J[c]}(t_1)} \,. \end{aligned}$$

The discount factor characterizes how much consumption the decision maker is willing to give up in period $t_1$ in order to receive an extra unit $\tilde{c}(t_2) = 1$ in period $t_2$. Therefore the discount factor is characterized by the expression

$$\frac{\widetilde{J[c]}(t_2)}{\widetilde{J[c]}(t_1)} \,.$$

Relating the general expression again to the standard model with intertemporally separable preferences (see equation 2) we find that, as is well known, the discount factor is simply characterized by the ratio of marginal utility in both periods.[1]

# 3 A Model of Habit Formation

In the following we analyze the consumption discount rate in a model where utility of current consumption depends on past consumption. Let $z(t) = \int_{-\infty}^{t} h(\tau,t)c(\tau)d\tau$ characterize a weighted average over past consumption. A special case is given by Ryder & Heal (1973), where $h(\tau,t) = \rho^h \exp(\rho^h(\tau - t))$. Define the welfare functional

$$J[c] = J[c, z[c]] = \int_0^\infty u\big(c(t), z(t)\big)\ \exp(-\rho t)\, dt\,. \tag{3}$$

We calculate the derivative of $J$ with respect to a change in consumption $\tilde{c}$ in appendix A to

$$\begin{aligned}\frac{d}{d\epsilon} J[c + \epsilon\, \tilde{c}]|_{\epsilon=0} &= \frac{d}{d\epsilon} \int_0^\infty \exp(-\rho t)\, u\big(c(t) + \epsilon\tilde{c}(t), z[c(t) + \epsilon\tilde{c}(t)]\big)\, dt \\ &= \int_{-\infty}^\infty \Big[ \exp(-\rho t)\, \mathbb{1}_{t\geq 0} u_1\big(c(t), z[c(t)]\big) \\ &\quad + \int_0^\infty \mathbb{1}_{t\leq\tau} h(t,\tau) \exp(-\rho\tau) u_2\big(c(\tau), z[c(\tau)]\big)\, d\tau \Big] \tilde{c}(t) dt\ .\end{aligned}$$

Comparing the latter expression with equation (1) yields

$$\begin{aligned}\widetilde{J[c]}(t) &= \mathbb{1}_{t\geq 0} \exp(-\rho t) u_1\Big(c(t), \int_{-\infty}^t h(\tau,t)c(\tau)d\tau\Big) \\ &\qquad + \int_0^\infty \mathbb{1}_{t\leq\tau} h(t,\tau) \exp(-\rho\tau) u_2\Big(c(\tau), \int_{-\infty}^t h(\tau,t)c(\tau)d\tau\Big)\, d\tau \\ &= \mathbb{1}_{t\geq 0} \exp(-\rho t) u_1(c(t), z(t)) + \int_0^\infty \mathbb{1}_{t\leq\tau} h(t,\tau) \exp(-\rho\tau) u_2\big(c(\tau), z(\tau)\big) d\tau\,.\end{aligned}$$

In what follows we will only be interested in calculating future welfare, implying $t \geq 0$ and

$$\widetilde{J[c]}(t) = \exp(-\rho t) u_1(c(t), z(t)) + \int_t^\infty h(t,\tau) \exp(-\rho\tau) u_2\big(c(\tau), z(\tau)\big) d\tau\,.$$

[1] An explicit time preference of utility as in the stationary discount utility model is part this marginal utility ratio.

The first term captures the immediate welfare change caused by a change of consumption at time $t$, while the second term captures the change of future welfare that is caused because of the (then historic) consumption change. In the Ryder & Heal (1973) example the density of the functional derivative, i.e. the discount factor, becomes

$$\widetilde{J[c]}(t) = \exp(-\rho t) u_1(c(t), z(t)) \\ + \rho^h \exp(\rho^h t) \int_t^\infty \exp(-(\rho + \rho^h)\tau) u_2\big(c(\tau), z(\tau)\big) d\tau \,.$$

# 4 The Social Discount Rate

## 4.1 Steady State

In this section we calculate the discount rate in a steady state for the Ryder & Heal (1973) setting. We assume that utility is derived in a Cobb Douglas form from absolute consumption and from consumption relative to the past. Moreover, intertemporal aggregation uses the constant elasticity of intertemporal substitution (CIES) form. These assumptions yield

$$u(c, z) = \frac{\left(c^\alpha \left(\frac{c}{z}\right)^{1-\alpha}\right)^{1-\sigma} - 1}{1-\sigma} = \frac{\left(c\, z^{-(1-\alpha)}\right)^{1-\sigma} - 1}{1-\sigma} \\ = \frac{\left(c^{1-\sigma} z^{-(1-\alpha)(1-\sigma)}\right) - 1}{1-\sigma} \,,$$

where $0 < \alpha < 1$ and $\sigma > 0$. The steady state assumption implies that $c(t) = c_0 \exp(gt)$ and $z(t) = z_0 \exp(gt)$ where $g \geq 0$ denotes the consumption growth rate and $z_0 = \frac{\rho^h}{\rho^h + g} c_0$.[2] The resulting discount factor becomes

$$\widetilde{J[c]}(t) = A(g) \exp\Big( - \big\{\rho + (1 - \alpha(1-\sigma))g\big\} t\Big) \,,$$

[2] For $g < 0$ the upcoming steady state calculation only holds if $\rho > -g(1 - \alpha(1-\sigma))\big\}$.

with $A(g) = \left[1 + (1-\alpha)\frac{\rho^h+g}{\rho+\rho^h+g-\alpha(1-\sigma)g}\right] c_0^{(1-\sigma)\alpha-1} \left(\frac{\rho^h}{\rho^h+g}\right)^{-(1-\alpha)(1-\sigma)}$. Therefore the discount rate in the steady state is

$$
\begin{aligned}
-\frac{\frac{d}{dt}\widetilde{J[c]}(t)}{\widetilde{J[c]}(t)} &= \rho + (1-\alpha(1-\sigma))g \\
&= \rho + \sigma g + (1-\alpha)(1-\sigma)g \qquad (4)\\
&= \rho + \sigma g \alpha + (1-\alpha)g \ .
\end{aligned}
$$

While the first two terms in equation (4) resemble the standard Ramsey equation for the consumption discount rate, the third term is new to the history dependent setup. For $\alpha = 1$ the history dependent term has no weight in the utility function and the term vanishes. For $\alpha = 0$ the history dependent term has full weight in the utility function. Then, the dependence of the discount rate on the intertemporal elasticity of substitution vanishes and $-\frac{\frac{d}{dt}\widetilde{J[c]}(t)}{\widetilde{J[c]}(t)} = \rho + g$. The corresponding decision maker appreciates current consumption only relative to past consumption. In a steady state this ratio $\frac{c(t)}{z(t)} = 1 + \frac{g_1}{\rho^h}$ is constant over time. In contrast, the dependence of the standard discount rate on the marginal elasticity of utility is driven by the fact that under growth future generations have a higher consumption level and, thus, a lower marginal utility from an extra unit of consumption.[3] Therefore in general, for an intertemporal elasticity of substitution greater than one ($\sigma < 1$), the history dependence increases the discount rate proportional to the growth rate. For an intertemporal elasticity of substitution smaller than one ($\sigma > 1$), the additional term decreases the discount rate proportional to the growth rate.

## 4.2 Anticipated Change of Growth

The interesting difference between the standard model and the history dependent model sets in when we analyze a change of the growth rate. In the standard setting, the change in the growth only effects the (instantaneous) social discount rate at times where the change in the growth rate occurs. In the history dependent setting, the social discount rate is affected already by an anticipated change in the growth rate as well as by a change of the growth rate in the past.

[3]However, intuition why g is there at all still to come ...

In the evaluation of climate change, an anticipated decline in the growth rate in the further future can reduce the effective discount rate already for the close future. Overall, more weight is given to anticipated future damages. The intuition is that "Getting used to the good life makes us more sensitive to lower welfare levels". However, at the same time, a catastrophic event that brings down the habit part of preferences would imply that we "get used to bad live". If the catastrophic event is followed again by growth that part makes the agent enjoy higher consumption more. A decision maker who is aware of habit formation has to balance these different effects of history dependence.

Calculations and formal results will follow.

# 5 Conclusions

We have derived a general approach to evaluate the marginal contribution of consumption when the value of an additional consumption unit depends on what has been consumed in previous or future periods. We have formulated a general model of habit formation and suggested a particularly intuitive and tractable form where welfare is a Cobb Douglas type mixture of absolute and relative present consumption with respect to a weighted averaged over past consumption. For such a preference specification we have analyzed the social discount rate in the steady state as well as for a stylized scenario with an anticipated decline of growth. In the steady state the discount rate simply gains a new component. Here the preference for smoothing consumption over time interacts with habit formation. Decision makers with a strong preference for smoothing consumption over time have a lower discount rate under habit formation. Decision makers who put less emphasis on smoothing consumption over time turn out to have a higher social discount rate as opposed to the standard model. An anticipated change in the growth rate has two effects. First, getting used to good life makes the evaluation more sensitive to a period of declining growth. Second, getting used to a relatively lower consumption level in a period of declining growth increases the valuation of an increasing consumption level after the decline. The balance of these two effects depends on the change in growth rate, the degree of habit formation and the degree of intertemporal substitutability of consumption. The detailed formal result will follow before the conference.

# A Functional Derivative for Habit Formation Model

In the following we calculate the derivative of $J$ with respect to a change in consumption $\tilde{c}$ for the habit formation model characterized by equation (3) in section 3:

$$
\begin{aligned}
\frac{d}{d\epsilon} J[c+\epsilon\,\tilde{c}]|_{\epsilon=0} &= \frac{d}{d\epsilon}\int_0^\infty \exp(-\rho t)\, u\big(c(t)+\epsilon\tilde{c}(t), z[c(t)+\epsilon\tilde{c}(t)]\big)\,dt \\
&= \frac{d}{d\epsilon}\int_0^\infty \exp(-\rho t)\Big[u\big(c(t), z[c(t)]\big) \\
&\qquad + u_1\big(c(t), z[c(t)]\big)\epsilon\tilde{c}(t) + u_2\big(c(t), z[c(t)]\big)\frac{d}{d\epsilon} z(c+\epsilon\tilde{c})\Big]\,dt\Big|_{\epsilon=0} \\
&= \frac{d}{d\epsilon}\int_0^\infty \exp(-\rho t) u_1\big(c(t), z[c(t)]\big)\epsilon\tilde{c}(t)\,dt\Big|_{\epsilon=0} \\
&\quad + \int_0^\infty \exp(-\rho t) u_2\big(c(t), z[c(t)]\big)\left[\frac{d}{d\epsilon}\int_{-\infty}^t h(\tau,t)\epsilon\tilde{c}(\tau)\,d\tau\right]_{\epsilon=0} dt \\
&= \int_0^\infty \exp(-\rho t) u_1\big(c(t), z[c(t)]\big)\tilde{c}(t)\,dt \\
&\quad + \int_0^\infty \exp(-\rho t) u_2\big(c(t), z[c(t)]\big)\left[\int_{-\infty}^\infty \mathbb{1}_{\tau\le t} h(\tau,t)\tilde{c}(\tau)\,d\tau\right] dt \\
&= \int_0^\infty \exp(-\rho t) u_1\big(c(t), z[c(t)]\big)\tilde{c}(t)\,dt \\
&\quad + \int_{-\infty}^\infty \tilde{c}(\tau)\left[\int_0^\infty h(\tau,t)\exp(-\rho t) u_2\big(c(t), z[c(t)]\big)\mathbb{1}_{\tau\le t}\,dt\right] d\tau \\
&= \int_{-\infty}^\infty \Big[\exp(-\rho t)\mathbb{1}_{t\ge 0} u_1\big(c(t), z[c(t)]\big) \\
&\quad + \int_0^\infty \mathbb{1}_{t\le\tau} h(t,\tau)\exp(-\rho\tau) u_2\big(c(\tau), z[c(\tau)]\big)\,d\tau\Big]\tilde{c}(t)\,dt\ .
\end{aligned}
$$

# B Relation to Hamiltonian Optimal Control

In the Ryder & Heal (1973) specification the model can be translated into a stock model, with $z(t)$ describing a 'habit stock'. The habit stock is formed by consumption in accordance to the equation of motion

$$\dot{z} = \rho^h (c - z)$$

and instantaneous welfare is a function of consumption and habit stock. Assume moreover, following Ryder & Heal (1973), a neoclassical growth model where saved consumption is invested in capital which again determines the productivity of a fixed amount of labor in the future. The equation of motion for the capital stock is given by

$$\dot{k} = f(k) - \gamma k - c \ , \tag{5}$$

where $\gamma$ is the depreciation rate of capital and $f$ is a neoclassical production function satisfying the standard assumptions. Then the optimization problem is described by the Hamiltonian

$$H(c, z, k, t) = u(c, z) \exp(-\rho t) + \lambda_z \rho^h (c - z) + \lambda_k (f(k) - \gamma k - c) \ .$$

Define the welfare function

$$W(t) = \int_0^t u\big(c(t), z(t)\big) \ \exp(-\rho t) \, dt \ ,$$

the extended state vector

$$x(t) = (x_0(t), x_1(t), x_2(t))' = (W(t), z(t), k(t))' \ ,$$

and the following vector characterizing the equations of motion for a given consumption path $c$

$$\begin{aligned} g(x) &= \big(g_0(x), g_1(x), g_2(x)\big)' \\ &= \big(u(c, z) \ \exp(-\rho t), \rho^h (c - z), f(k) - \gamma k - c\big)' \ . \end{aligned}$$

Then the equations of motion, including the time development of welfare, are summarized by

$$\dot{x}(t) = g(x) x(t) \ . \tag{6}$$

In order to capture the welfare effect of an additional unit of consumption we have to consider its direct welfare effect as well as the indirect or stock effect on welfare caused by habit formation. This is obtained by analyzing an exogeneous consumption change that keeps the capital stock unchanged.[4] Let the pair $(c, z)$ denote an admissible path.

[4] We calculate the welfare effect of an additional unit of consumption in period $t$. In the optimum,

To perturb $c$ by a positive constant $\Delta c$ on a time interval $A = (t_0 - \epsilon, t_0)$ define the new consumption path $c^p$ by

$$c^p(t) = \begin{cases} c(t) + \Delta c \text{ for } t \in A \\ c(t) \text{ for } t \notin A \end{cases}$$

Such a perturbation affects welfare directly and through the habit formation 'stock'. Precisely it perturbs the stock variables $x_0(t) = W(t)$ and $x_1(t) = z(t)$ around time $t_0$. By (the exogeneity) assumption the capital stock $x_2(t) = k(t)$ is unaffected. The change in $z(t_0)$ is given by

$$\begin{aligned} \Delta z = z^p(t_0) - z(t_0) &= \big[g_1\big(c^p(t_0), z(t_0)\big) - g_1\big(c(t_0), z(t_0)\big)\big]\,\epsilon + o(\epsilon) \\ &= \frac{\partial g_1\big(c, z\big)}{\partial c}(t_0)\Delta c\,\epsilon + o(\epsilon\,\Delta c)\ , \end{aligned}$$

where the Landau symbol $o(\epsilon\,\Delta c)$ denotes changes in the order higher than $\Delta c$ times $\epsilon$. The change of $x_0$ at time $t_0$ is found to be[5]

$$\begin{aligned} \Delta W_{\Delta c,\epsilon}(t_0) &= W(c^p, z^p, t_0) - W(c, z, t_0) \\ &= \int_{t_0-\epsilon}^{t_0} \big[u\big(c^p(\tau), z^p(\tau)\big) - u\big(c(\tau), z(\tau)\big)\big]\ \exp(-\rho\tau)\, d\tau \\ &= \left[\frac{\partial u\big(c(t_0), z(t_0)\big)}{\partial c(t_0)}\Delta c + o(\Delta c)\right. \\ &\qquad \left. + \frac{\partial u\big(c(t_0), z(t_0)\big)}{\partial z(t_0)}\,\frac{\partial g_1\big(c(t_0), z(t_0)\big)}{\partial c}\Delta c\,\epsilon + o(\epsilon\,\Delta c)\right]\,\epsilon\ \exp(-\rho t) \\ &= \frac{\partial u\big(c(t_0), z(t_0)\big)}{\partial c(t_0)}\Delta c\,\epsilon\ \exp(-\rho t_0) + o(\epsilon\,\Delta c) \end{aligned}$$

however, admissible consumption paths satisfying the equation of motion (5) yield welfare effects that level out to zero and give no information on the individual welfare changes caused by consumption changes at different times. Control perturbations are part of the (non-variational) proof of Pontryagin, Boltyanskii, Gamkrelidze & Mishchenko's (1962, 86 et sqq) maximum principle. For a more accessible presentation see e.g. Beavis & Dobbs (1990, 310 et sqq).

[5] The approximation of the integral in $\epsilon$ and $\Delta c$ seems to be related to the assumption that $t$ is a Lesbesgue or regular point, see Lee & Markus (1967, 248).

Now we analyze how an $\epsilon$-change of $x$ at time $t_0$ affects the states $x(t)$ at later times. Hereto note that for $t > t_0$ equation (6) has to be satisfied for the perturbed system for all perturbed paths $x_\epsilon(t)$. Taking the derivative with respect to the effect of an $\epsilon$-change on both sides of equation (6) yields[6]

$$\frac{\partial}{\partial \epsilon}\dot{x} = \frac{\partial g(x)}{\partial x}\frac{\partial x}{\partial \epsilon}$$

$$\Leftrightarrow \frac{d}{d\tau}\frac{\partial x}{\partial \epsilon} = \frac{\partial g(x)}{\partial x}\frac{\partial x}{\partial \epsilon} \tag{7}$$

The rearrangement is possible because (under adequate regularity conditions) the solution of the differential equation (7) is continuously differentiable in $\epsilon$. The differential equation (7) is the adjoint to the equation of motion of the costate vector $\lambda = (1, \lambda_z, \lambda_k)$, where the constant shadow value of welfare is normalized to $\lambda_W = 1$. Let $\Psi(t, t_0)$ be the fundamental solution of the adjoint system, i.e. a time dependent $3 \times 3$ matrix that satisfies

$$\lambda(t) = \Psi(t, t_0)\lambda(t_0) \ .$$

In particular, for $\lambda(t) = (1, 0, 0)$, the relation

$$\begin{aligned}\lambda(t_0) &= \Psi(t, t_0)^{-1}\lambda(t) = \Psi(t_0, t)(1, 0, 0)' \\ &= \big(\Psi_{11}(t_0, t), \Psi_{12}(t_0, t), \Psi_{13}(t_0, t)\big)'\end{aligned}$$

holds. It is a well known fact that the fundamental solution for $\frac{\partial x}{\partial \epsilon}$ in equation (7) is the transpose of the inverse of the fundamental solution to the adjoint system, see e.g. Athans & Falb (1966, 147 et seqq). Therefore we have

$$\frac{\partial x}{\partial \epsilon}(t) = \Psi(t_0, t)'\frac{\partial x}{\partial \epsilon}(t_0)$$

and, in particular, the change of welfare at time $t > t_0$ caused by a change of the state variables at $t_0$ is given by

$$\begin{aligned}\frac{\partial}{\partial \epsilon}W(t) = \frac{\partial x_0}{\partial \epsilon}(t) &= \big(\Psi_{11}(t_0, t), \Psi_{12}(t_0, t), \Psi_{13}(t_0, t)\big)\frac{\partial x}{\partial \epsilon}(t_0) \\ &= \frac{\partial x_0}{\partial \epsilon}(t_0) + \lambda_z(t_0)\frac{\partial z}{\partial \epsilon}(t_0) + \lambda_k(t_0)\frac{\partial k}{\partial \epsilon}(t_0) \ .\end{aligned}$$

In consequence, the welfare effect at time $t$ of the exogenous consumption perturbation is

[6]The analysis describes how a tangent vector is carried along the curve $x(t)$.

in first order of $\epsilon$ and $\Delta c$

$$\Delta W_{\Delta c,\epsilon}(t) = \frac{\partial x_0}{\partial \epsilon}(t)\epsilon = \frac{\partial u\big(c(t_0), z(t_0)\big)}{\partial c(t_0)}\Delta c\, \epsilon \exp(-\rho t_0) + \lambda_z(t_0)\frac{\partial g_1\big(c(t_0), z(t_0)\big)}{\partial c}\Delta c\, \epsilon$$

Not that the expression is independent of $t$. The time horizon is taken care of by the value of $\lambda(t_0)$.
The effect of a change of an 'infinitesimal unit' of consumption in an infinitesimal time interval around $t_0$ is therefore given by the density function

$$\lim_{\Delta c \to 0,\, \epsilon \to 0} \frac{W_{\Delta c,\epsilon}(t)}{\Delta c\, \epsilon} = \frac{\partial u\big(c(t_0), z(t_0)\big)}{\partial c(t_0)} \exp(-\rho t_0) + \frac{\partial g_1\big(c(t_0), z(t_0)\big)}{\partial c}\lambda_z(t_0) \tag{8}$$

The equation of motion for the habit stock shadow value is

$$\dot{\lambda}_z = -\left[\exp(-\rho t)\frac{\partial u(c,z)}{\partial z} - \lambda_z \rho^h\right] .$$

The solution to the homogenous equation is

$$\lambda_z(t) = A \exp(\rho^h t) .$$

Using variation of the constant, we find

$$\dot{A}(t)\exp(\rho^h t) + A(t)\rho^h \exp(\rho^h t) = -\exp(-\rho t)\frac{\partial u(c,z)}{\partial z} + \rho^h A(t)\exp(\rho^h t)$$

$$\Leftrightarrow \dot{A}(t)\exp(\rho^h t) = -\exp(-\rho t)\frac{\partial u(c,z)}{\partial z}$$

$$\Leftrightarrow \dot{A}(t) = -\exp(-(\rho + \rho^h)t)\frac{\partial u(c,z)}{\partial z}$$

$$\Leftrightarrow A(t) = A_0 - \int_{t_0}^{t} \frac{\partial u(c,z)}{\partial z} \exp(-(\rho+\rho^h)\tau)d\tau ,$$

and therefore

$$\lambda_z(t) = A_0 \exp(\rho^h t) - \exp(\rho^h t)\int_{t_0}^{t} \frac{\partial u(c,z)}{\partial z} \exp(-(\rho+\rho^h)\tau)d\tau$$

To determine the integration constant $A_0$, we analyze the transversality condition

$$0 = \lim_{t\to\infty} \lambda_z(t) = \lim_{t\to\infty} \exp(\rho^h t) \left[ A_0 - \int_{t_0}^{t} \frac{\partial u(c,z)}{\partial z} \exp(-(\rho+\rho^h)\tau) d\tau \right]. \tag{9}$$

A necessary condition for equation (9) to hold is that

$$A_0 = \int_{t_0}^{\infty} \frac{\partial u(c,z)}{\partial z} \exp(-(\rho+\rho^h)\tau) d\tau$$

and thus

$$\lambda_z(t) = \exp(\rho^h t) \int_{t}^{\infty} \frac{\partial u(c,z)}{\partial z} \exp(-(\rho+\rho^h)\tau) d\tau \ . \tag{10}$$

Inserting expression (10) for the shadow value into equation (8) and replacing $t_0$ by $t$ yields the density

$$\frac{\partial u\big(c(t), z(t)\big)}{\partial c(t)} \exp(-\rho t) + \rho^h \exp(\rho^h t) \int_{t}^{\infty} \frac{\partial u(c,z)}{\partial z} \exp(-(\rho+\rho^h)\tau) d\tau \ ,$$

which, as it should, coincides with $\widetilde{J[c]}(t)$ in equation (3).

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