Intertemporal Risk Aversion, Stationarity, and Discounting

Christian Traeger

Department of Agricultural & Resource Economics,
UC Berkeley

- Introduce a more general preference representation & risk attitude
- Add axioms characterizing what to keep from standard model
- Derive implications for social discounting

Motivation: Climate Change & Optimal Greenhouse Gas Abatement

Crucial modeling determinants:

- ▷ Uncertainty
- ▷ Pure time preference (impatience)

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Paper points out that the discounted expected utility (*standard*) model

- ▷ contains an implicit assumption of risk neutrality

and that in the more general model

- ▷ widespread constraints on decision making under uncertainty imply a zero rate of pure time preference

Outline:

- ▷ Upfront summary
- ▷ Some related literature
- ▷ General representation under
  Certainty stationarity and additivity + *von Neumann-Morgenstern axioms*
- ▷ Intertemporal risk attitude:
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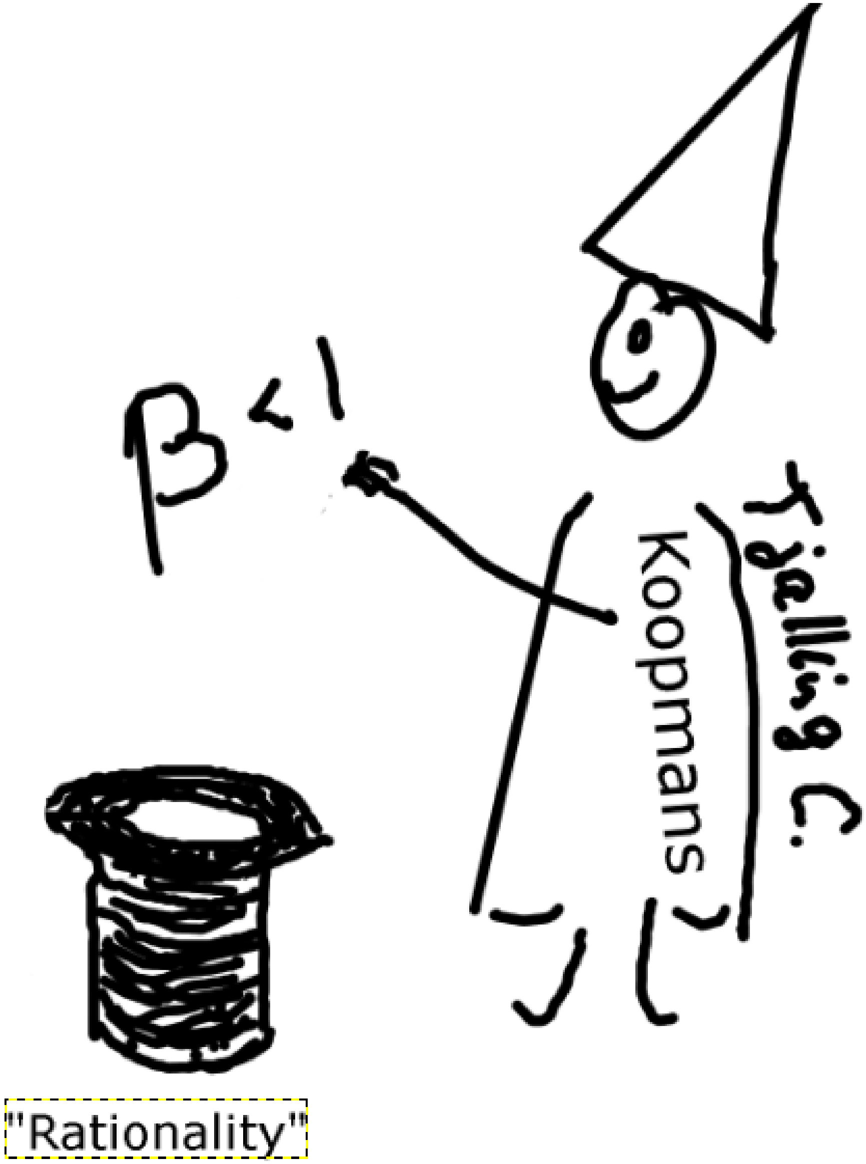
- ▷ Add axioms of
  Risk stationarity + *indifference to the timing of risk resolution*
- ▷ Results on discounting
- ▷ Conclusions

Upfront Summary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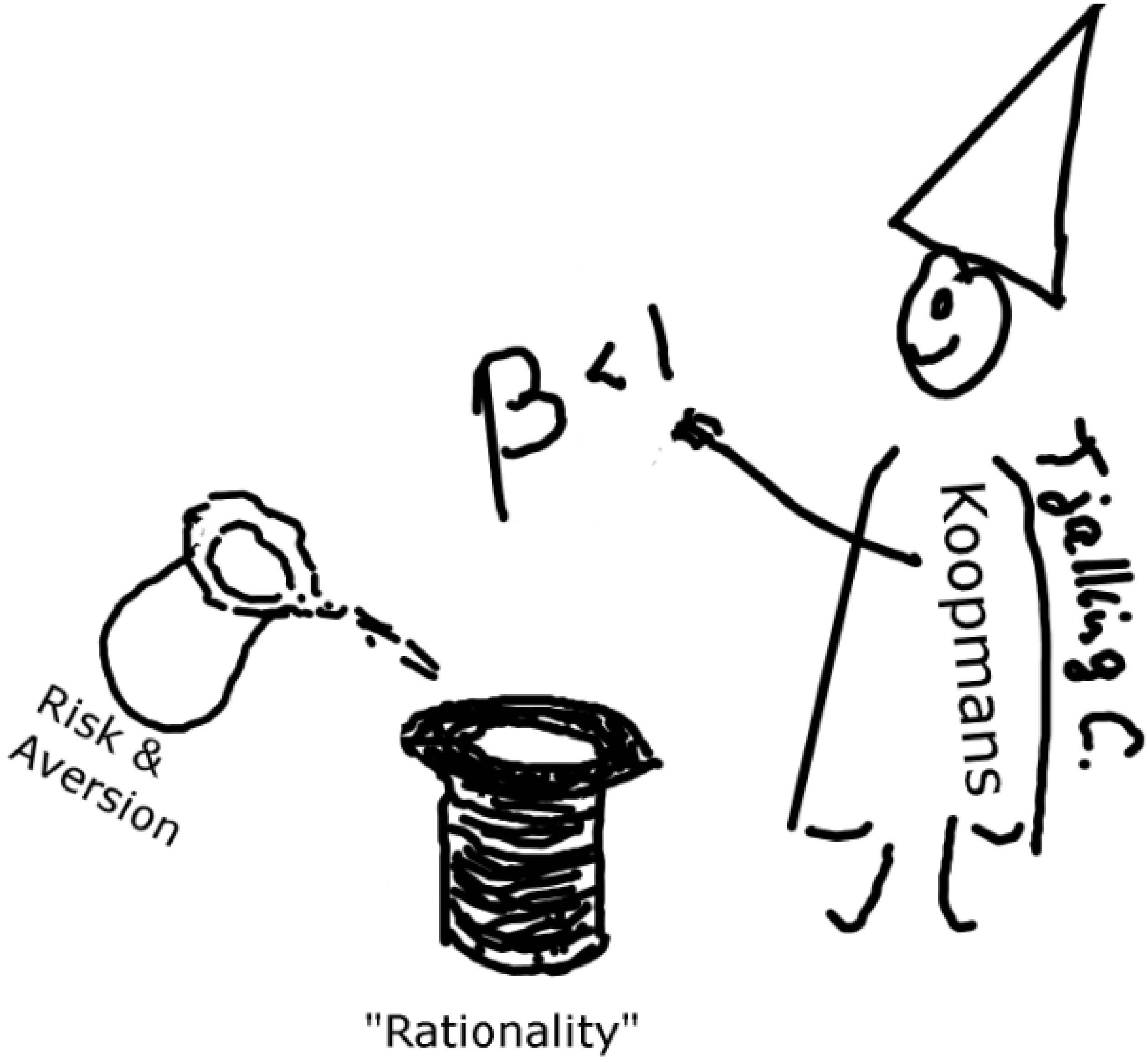
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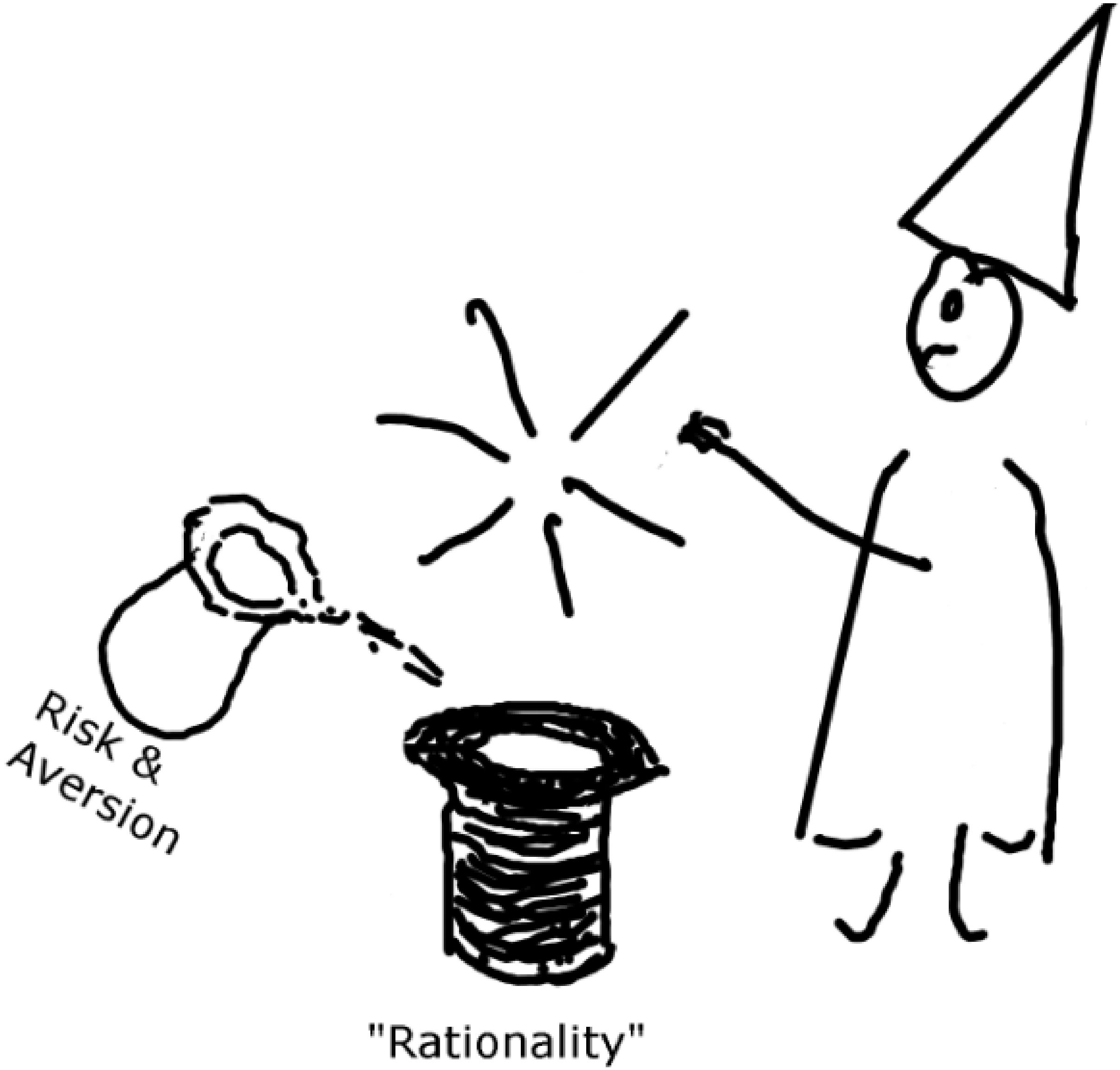

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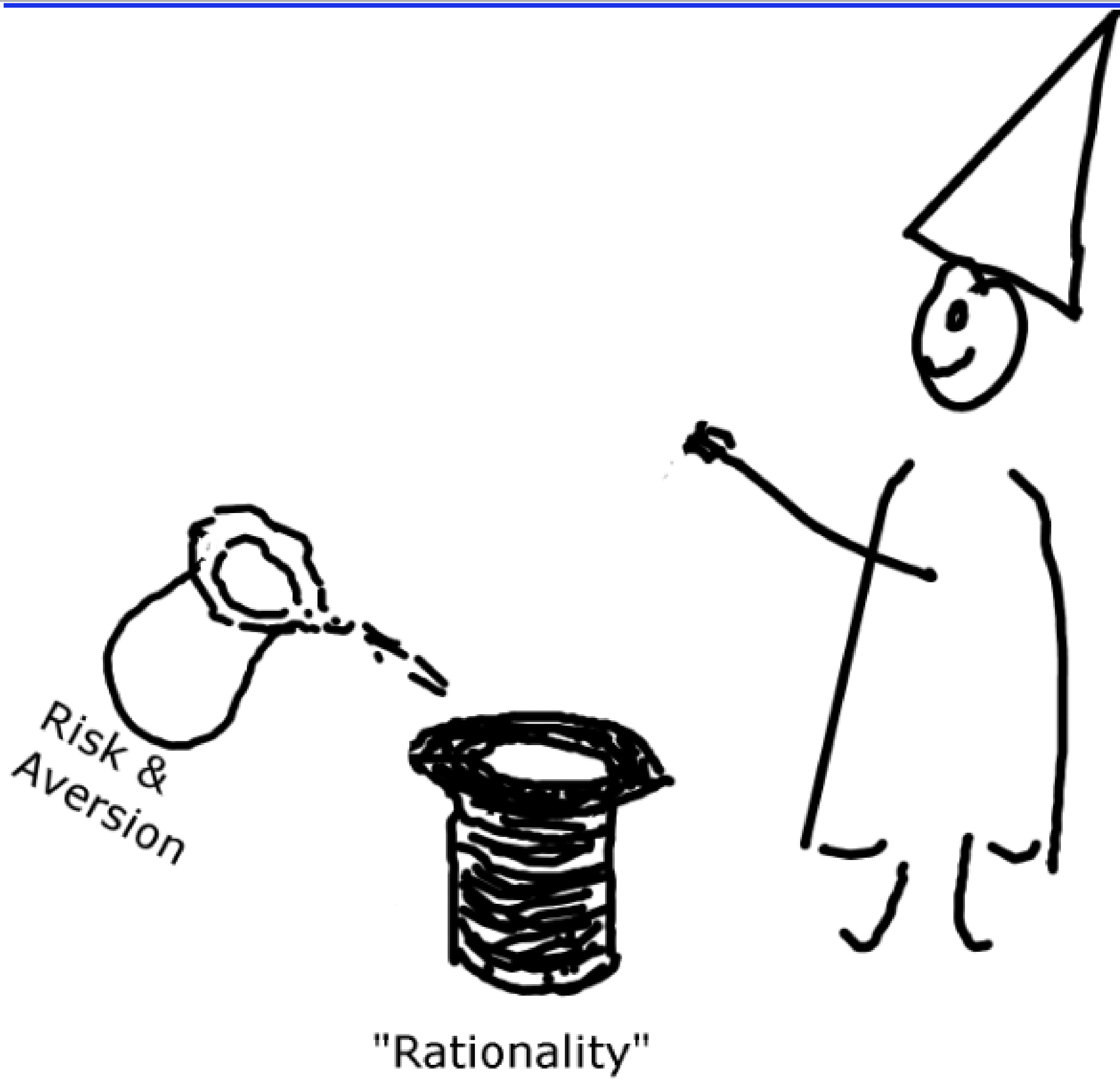

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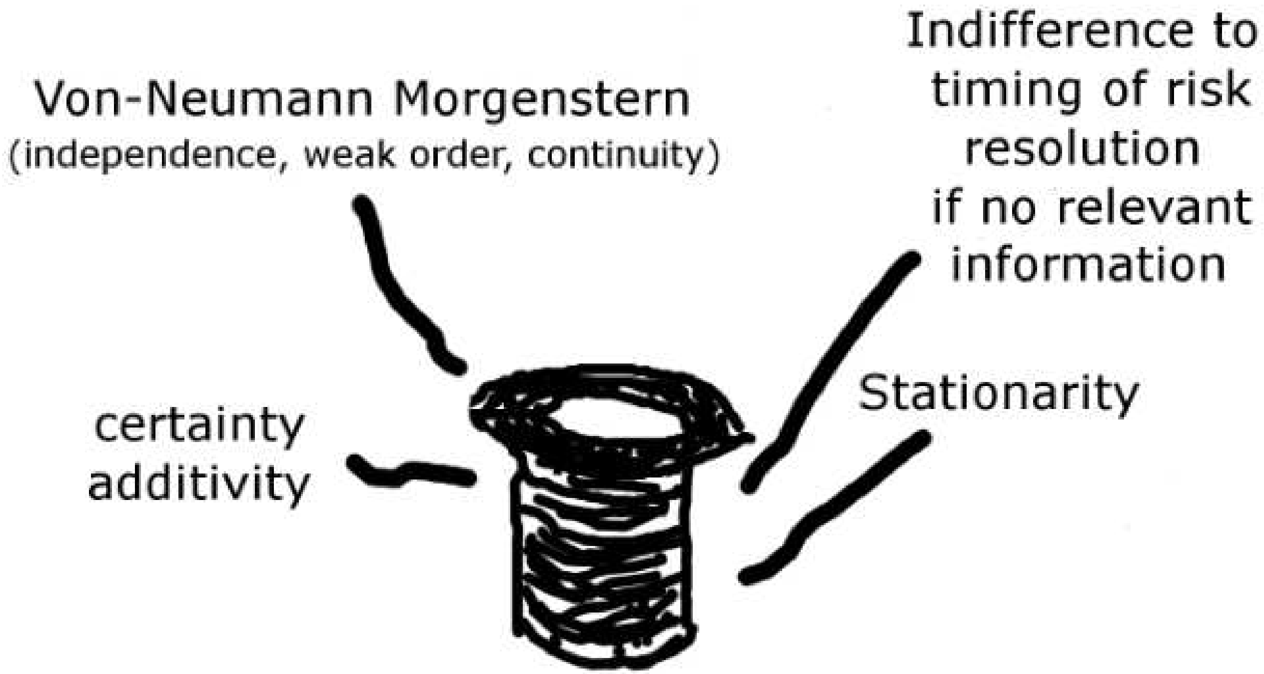

Epstein & Zin (1989,E): Disentangle Arrow Pratt risk aversion from intertemporal elasticity of substitution

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The time and uncertainty additive (expected utility) standard model:

$$\left.\begin{array}{lccc} \text{Time additive} & : \sum_t & u_t & (x_t) \\ \text{Expected utility} & : \text{E} & u_t & (x_t) \end{array}\right\} \text{E} \sum_t \; u_t \; (x_t)$$

Remark:

For stationary preferences: $u_t = \beta^t u$

$\beta = \frac{1}{1+\rho}$: utility discount factor

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$\hookrightarrow$ We will end up needing two functions!

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As opposed to the above authors:

- ▷ General preferences satisfying vNM axioms
- ▷ Introduce axiomatic definition of (multi-commodity) intertemporal risk aversion
- ▷ Analyze axiomatic consequences for discounting

Koopmans (1960,E): The great wizard...

Kreps & Porteus (1978,E):

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This paper:

- ▷ Start from certainty additive framework
  - ▷ Can preserve linearity over time
  - ▷ But requires non-linear risk aggregation
- ▷ Discuss intertemporal risk aversion, stationarity, discounting
- ▷ Remark: Kreps & Porteus's (1978) timing preference can be explained by a change intertemporal risk attitude over time

Setup - The Choice Space

- Time: discrete, arbitrary finite planning horizon T
- X : Space of goods (outcomes) *(connected compact metric)*
- $\Delta(\cdot)$: Set of Borel *probability* measures on space '$\cdot$' *(Prohorov metric)*

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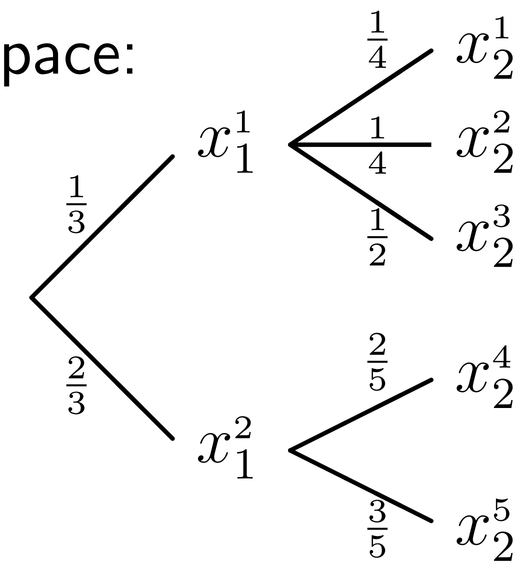

General $\cdots\ \Delta\ \Big(X \times \Delta\ (X)\Big)$: Temporal Lottery (rather than $\Delta(X \times X)$)

P_{T-1} (for the outer Δ), P_T (for the inner Δ) : Choice Spaces in Periods $T-1$, T

$\succeq_{T-1}$ $\succeq_T$: Preferences in Periods $T-1$, T

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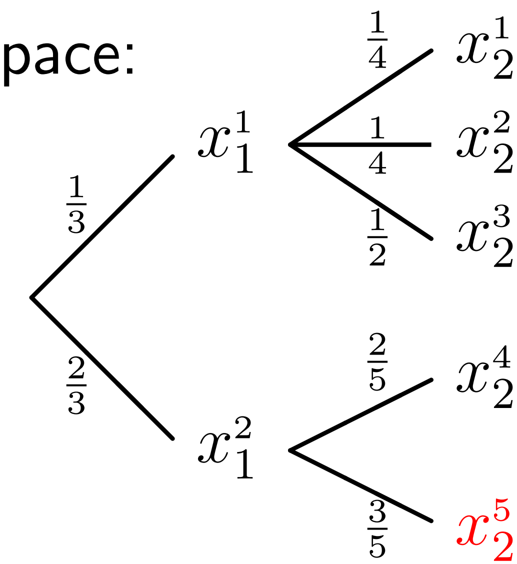

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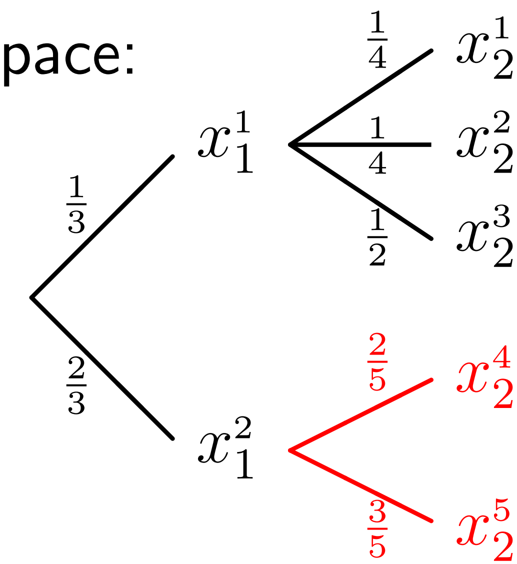

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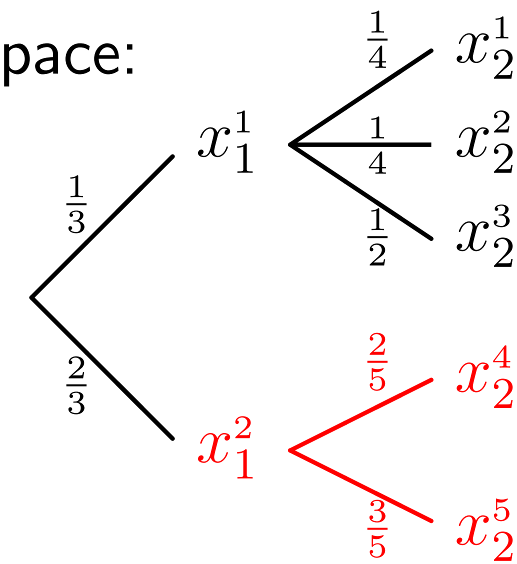

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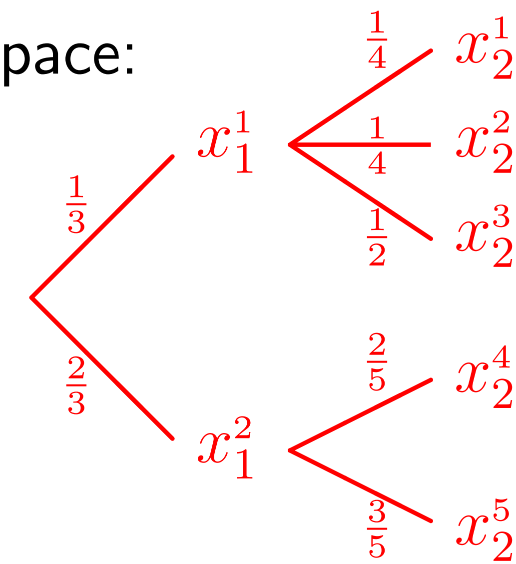

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Notation: Given $\mathrm{x} \in \mathrm{X}^t$ define

- $(\mathrm{x}_{-\tau}, x) = (\mathrm{x}_t, ..., \mathrm{x}_{\tau-1}, x, \mathrm{x}_{\tau+1}, ..., \mathrm{x}_T) \in \mathrm{X}^t$: consumption path coinciding with x in all but the τ^{th} period, in which it yields outcome x.

Notation: For a (compact metric) space Y define

- $\mathcal{C}^0(Y)$: Set of continuous functions $Y \to \mathbb{R}$

Axioms:

A1-A3 (vNM axioms) Standard, applied for $\succeq_t$ on P_t with $t \in \{1, ..., T\}$

- (*weak order*) $\succeq_t$ is transitive and complete
- (*independence*) $\forall\, p, q, r \in P_t$:
  $p \sim_t q \quad \Rightarrow \quad \lambda p + (1-\lambda) r \sim_t \lambda q + (1-\lambda) r \quad \forall\, \lambda \in [0,1]$
- (*continuity*) $\forall p \in P_t$:
  The sets $\{q \in P_t : q \succeq_t p\}$ and $\{q \in P_t : p \succeq_t q\}$ are closed in P_t

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A4 (certainty separability)

For all $\mathrm{x}, \mathrm{x}' \in \mathrm{X}^1$, $x, x' \in X$ and $\tau \in \{1, ..., T\}$ it holds that

$i)$ $(\mathrm{x}_{-\tau}, x) \succeq_1 (\mathrm{x}'_{-\tau}, x) \Leftrightarrow (\mathrm{x}_{-\tau}, x') \succeq_1 (\mathrm{x}'_{-\tau}, x')$

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A5 (*time consistency*)

For all $t \in \{1, ..., T-1\}$, $x_t \in X$ and $p_{t+1}, p'_{t+1} \in P_{t+1}$:

$$(x_t, p_{t+1}) \succeq_t (x_t, p'_{t+1}) \quad \Leftrightarrow \quad p_{t+1} \succeq_{t+1} p'_{t+1}.$$

Definition: A decision maker's preferences are called

certainty stationary...

- *graphical 2 period illustration:*

  ... iff for all $\overline{x}, \underline{x}, x^*, x \in X$

$$\overline{x} \text{ —— } x \quad \succeq_1 \quad x^* \text{ —— } x \quad \Leftrightarrow \quad \overline{x} \quad \succeq_2 \quad x^*$$

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For $f : \mathbb{R} \to \mathbb{R}$ continuous and strictly increasing and some compact metric space Y define

- $\mathcal{M}^f : \Delta(Y) \times \mathcal{C}^0(Y) \to \mathbb{R}$
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It satisfies

- ▷ $\mathcal{M}^f(y, u) = u(y) \ \forall \, y \in Y$ (degenerate lottery)
- ▷ $\mathcal{M}^f(p, u) = \mathrm{E}_p \, u$ for $f = \mathrm{id}$ (expected value)

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Includes rules

- 'CRRA form': $f = \mathrm{id}^\alpha$ (related to Epstein-Zin)
- 'ARRA form': $f = \exp^\xi$ ($f(z) = \exp^{\xi z}$, will come up here)

Example

Theorem 1:

A preference relation $\succeq \equiv (\succeq_t)_{t\in\{1,...,T\}}$ on $(P_t)_{t\in\{1,...,T\}}$ satisfies

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there exists $\beta \in \mathbb{R}_{++}$, a continuous function $u : X \to \mathbb{R}$, and strictly increasing and continuous functions $f_t : \mathbb{R} \to \mathbb{R}$ for $t \in \{1, ..., T\}$ such that defining $\tilde{u}_T(x_T) = u(x_T)$ and recursively

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$$\Big(\text{i.e. } p_t \ \succeq_t \ p'_t \quad \Leftrightarrow \quad \mathcal{M}^{f_t}(p_t, \tilde{u}_t) \ \geq \ \mathcal{M}^{f_t}(p'_t, \tilde{u}_t) \quad \forall p_t, p'_t \in P_t\,.\Big)$$

(Illustration)

Let x, x' be two consumption paths of length T.

Example, $T = 4$:

$$x = (\quad,\quad,\quad,\quad)$$

$$x' = (\quad,\quad,\quad,\quad)$$

Let $x \succ x'$ denote a strict preference for x over x'.
Let $\sim$ denote indifference.

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Example, $T = 4$:

$$x = (☺, ☹, ☺, ☹)$$

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Assume you'd be indifferent between

$$(\smile,\frown,\smile,\frown) \sim (\frown,\smile,\frown,\smile)$$

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(☺, ☹, ☺, ☹) ∼ (☹, ☺, ☹, ☺)

If not, please mentally adjust the corners of the mouth of the red frowny ☹ to reach indifference.

Assume you'd be indifferent between

$$(\smiley,\frownie,\smiley,\frownie) \sim (\frownie,\smiley,\textcolor{red}{\frownie},\smiley)$$

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What preference do you have in the following choice?

$$(\smiley,\frownie,\smiley,\frownie) \text{ vs. } \begin{cases} \frac{1}{2} & (\smiley,\smiley,\smiley,\smiley) \\ \frac{1}{2} & (\frownie,\frownie,\textcolor{red}{\frownie},\frownie) \end{cases}$$

certain path coin toss lottery

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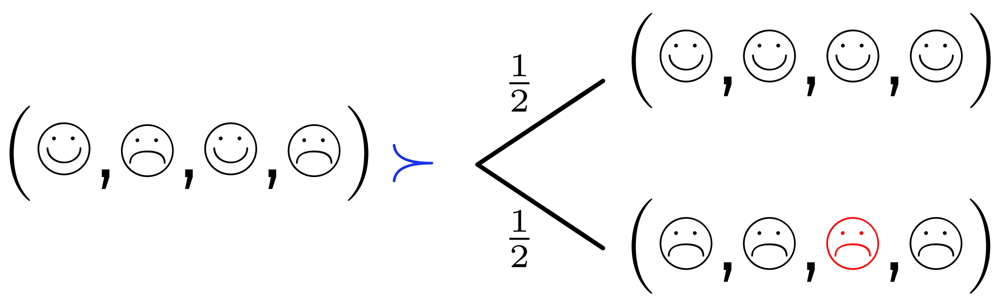

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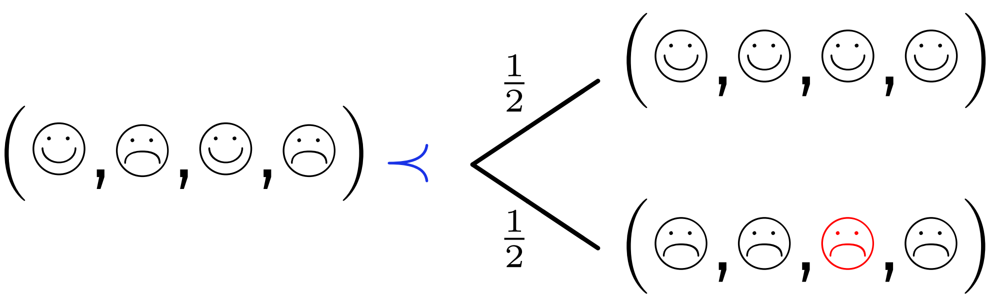

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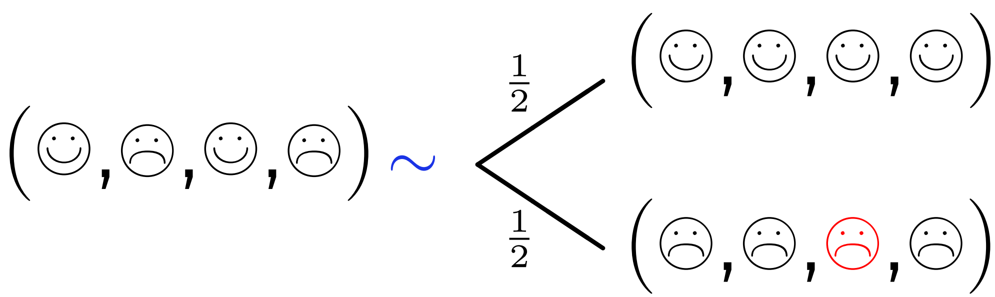

certain path coin toss lottery

STANDARD MODEL $\mathrm{E}\sum_t \beta^t u(x_t)$

Assume you'd be indifferent between

$$(\smiley,\frownie,\smiley,\frownie) \sim (\frownie,\smiley,\frownie,\smiley)$$

If not, please mentally adjust the corners of the mouth of the red frowny $\frownie$ to reach indifference.

What preference do you have in the following choice?

$$(\smiley,\frownie,\smiley,\frownie) \succ \begin{cases} \frac{1}{2} & (\smiley,\smiley,\smiley,\smiley) \\ \frac{1}{2} & (\frownie,\frownie,\frownie,\frownie) \end{cases}$$

certain path coin toss lottery

INTERTEMPORAL RISK AVERSE DM

Theorem 2:

In the representation of theorem 1, a decision maker is strictly intertemporal risk averse in period $t < T$, if and only if, f_t is strictly concave.

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$\rightarrow$ Time additive expected utility standard model

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Interpretation:

f_t measures risk aversion with respect to utility gains and losses

$$\mathcal{M}^{f_t}(p_t, \tilde{u}_t) = {f_t}^{-1}\Big(\underline{\mathsf{E}_{p_t}\, f_t\big[u(x_t)}+\beta\mathcal{M}^{f_{t+1}}(p_{t+1},\tilde{u}_{t+1})\big]\Big)$$

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Define the measure of absolute intertemporal risk aversion as

$$\mathrm{AIRA}_t(z) = -\frac{\frac{d^2}{dz^2} f_t(z)}{\frac{d}{dz} f_t(z)} \, .$$

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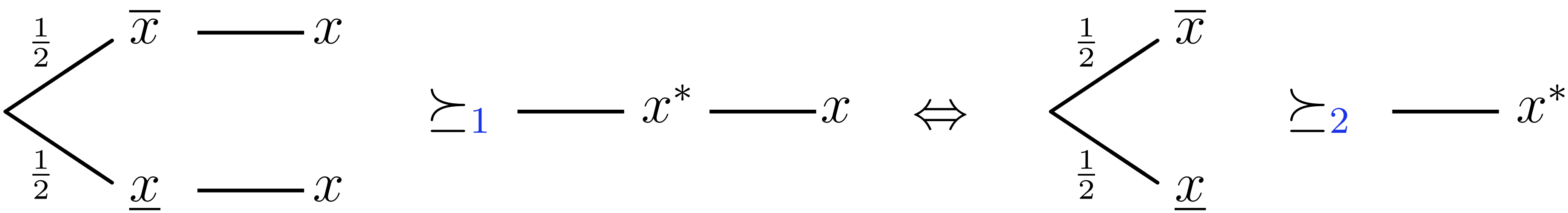
Note: f_t and AIRA_t measure risk aversion with respect to current value welfare gains and losses in period t.

Definition: A decision maker's preferences are called risk stationary...

- *graphical 2 period illustration:*

  ... iff for all $\overline{x}, \underline{x}, x^*, x \in X$



- *general definition:*

  ...iff for all $t \in \{1, ..., T-1\}$ and $x \in X$:

$$\frac{1}{2}(\mathrm{x}, x) + \frac{1}{2}(\mathrm{x}', x) \succeq_t (\mathrm{x}'', x) \quad \Leftrightarrow \quad \frac{1}{2}\mathrm{x} + \frac{1}{2}\mathrm{x}' \succeq_{t+1} \mathrm{x}''$$

  for all $\mathrm{x}, \mathrm{x}', \mathrm{x}'' \in \mathrm{X}^{t+1}$.

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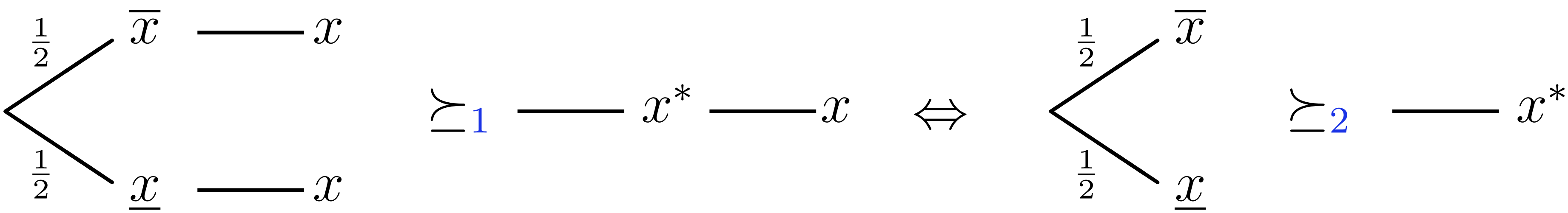

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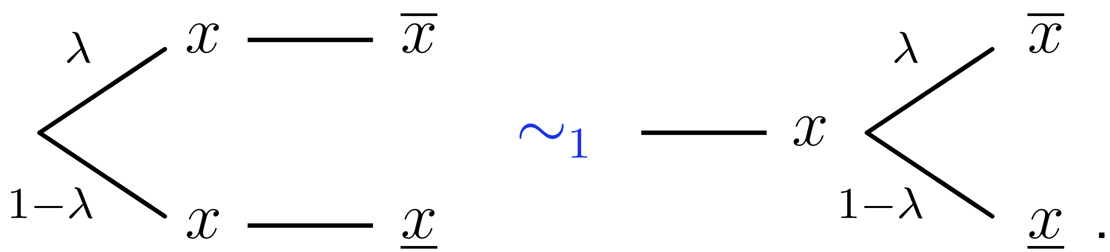
Definition: A decision maker is timing indifferent...

- *graphical 2 period intuition:*

  ...iff for *all* outcomes $x \in X$ and for all $\overline{x}, \underline{x} \in X, \lambda \in [0,1]$



  Uncertainty resolves ('biased coin toss takes place') in
  first versus second period

- *general definition:*

  ...iff for all $t \in \{1, ...T-1\}$ and *all* $x_t \in X$,
  and for all $p_{t+1}, p'_{t+1} \in P_{t+1}$ and all $\lambda \in [0,1]$ holds

$$\lambda(x_t, p_{t+1}) + (1-\lambda)(x_t, p'_{t+1}) \quad \succeq_t \quad (x_t, \lambda p_{t+1} + (1-\lambda)p'_{t+1}).$$

Define $p_t^{\mathrm{x}} \in \Delta(\mathrm{X}^t)$ as the non-recursive lottery obtained from p_t by 'integrating out' the information on timing of risk resolution

Define $p_t^{\mathcal{X}} \in \Delta(\mathrm{X}^t)$ as the non-recursive lottery obtained from p_t

Theorem 3:

A sequence of binary relations $(\succeq_t)_{t\in\{1,...,T\}}$ on $(P_t)_{t\in\{1,...,T\}}$ satisfies

$i)$ vNM axioms, additive separability, time consistency

$ii)$ indifference to the timing of risk resolution

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if and only if, there exists a continuous function $u : X \to \mathbb{R}$,
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Note: $\mathcal{M}^{\exp^0}(p_t^{\mathrm{x}}, \tilde{u}_t) \equiv \lim_{\xi\to 0} \mathcal{M}^{\exp^\xi}(p_t^{\mathrm{x}}, \tilde{u}_t) = \mathsf{E}_{p_t^{\mathrm{x}}} \sum_{\tau=t}^{T} \beta^{\tau-1} u(\mathrm{x}_\tau^t)$

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Intuition

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Risk Stationarity

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Intertemporal risk aversion can only be constant over time in present and in current value, if

- ▷ it is zero $\Rightarrow$ intertemporal risk neutral standard model
- ▷ both values coincide nontrivially $\Rightarrow$ no pure time preference

An intertemporal risk averse decision maker discounts
for reasons of uncertainty:

- Consider a representation in the sense of theorem 3 with $\beta = 1$
- Assume for simplicity that risk is independent between periods: $p_t^{\mathsf{X}} = \mathsf{p}_1 \otimes \mathsf{p}_2 ... \otimes \mathsf{p}_T$ where $\mathsf{p}_t \in \Delta(X)$ describes risk over x_t

Then the evaluation simplifies to

$$\mathcal{M}^{\exp^\xi}(p_t^{\mathsf{X}}, \tilde{u}_t) = \mathcal{M}^{\exp^\xi}(\mathsf{p}_1, u) + \mathcal{M}^{\exp^\xi}(\mathsf{p}_2, u) + ... + \mathcal{M}^{\exp^\xi}(\mathsf{p}_T, u) \quad (*)$$

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- ▷ Let expected per period utility be constant over time: $\mathsf{E}\, u(x_t) = \bar{u}$ for all $t \in \{1, ..., T\}$
- ▷ Let uncertainty over utility increase over time: Strictly more weight on the tails of the induced utility lottery

Then the summands in equation $(*)$ decrease over time, i.e.

$$\mathcal{M}^{\exp^\xi}(\mathsf{p}_{t+1}, u) < \mathcal{M}^{\exp^\xi}(\mathsf{p}_t, u)$$

conclusions evidence

Relating these findings to climate change evaluation implies:

- ▷ The Stern review argues for a zero rate of pure time preference based on ethical arguments
- ↪ These ethical arguments are critizised as a British utilitarian perspective
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- ▷ We found: also simple constraints on decision making under uncertainty can lead to a zero rate of pure time preference
- ↪ Pay more attention to the long-run:
  The more we know about climate change, the more attention we should also pay to its long-term effects
- ↪ Pay more attention to reducing risk

Related empirics

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- Intertemporal risk averse agents discount future utility for reasons of increasing uncertainty

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