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On the Optimal Allocation of Students and Resources in a System of Higher Education*

James M. Sallee, Alexandra M. Resch, and Paul N. Courant

Abstract

We model the social planner's decision to establish universities and populate them with students and resources, given a distribution of student ability and a limited pool of resources for higher education. If student ability and school resources are complements, and if there is a fixed cost to establishing a school, then the optimal allocation will involve a tiered system of higher education that sorts students by ability. In contrast to previous research, we show this tiered system is optimal even in the absence of peer effects. In considering where to locate students, the planner balances the benefit of providing students with more resources against the congestion costs of overcrowding schools. Nearly identical students who are close to the margin of entry to a higher or lower tier will experience discrete gaps in education quality. In considering how many universities to establish, the planner will balance the value of more precise tailoring against the cost of establishing additional schools. The planner's inability to perfectly tailor education quality will result in both winners and losers. Our model also makes predictions about how university systems that serve different populations should vary. Larger systems will produce more per dollar of expenditures and more education per student, due to economies of scale.

KEYWORDS: economics of higher education, public higher education

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1 Introduction

Approximately 77% of college students in the United States attend public institutions, where total annual expenditures now exceed $190 billion (NCES, 2005a,b). Despite the magnitude of public involvement in higher education, and despite the enormous body of research on the economics of education, economists have not established a normative model of how students and resources should be allocated in a system of public higher education. The aim of this paper is to provide a simple and tractable framework for the analysis of this issue and to derive several intuitive results regarding the optimal allocation of students and resources.

Given a distribution of student ability and a limited pool of resources for higher education, we model the social planner's decision to establish schools and populate them with students and resources. Our model is driven by two simple assumptions: (1) that there is complementarity between resources and student ability in producing educational outcomes; and (2) that there is a fixed cost to establishing a school. We show that these assumptions produce a tiered structure that sorts students by skill and results in discontinuous spending and educational output per student for essentially identical students at the margin between schools.

The existence of a fixed cost creates economies of scale, both for individual schools and for the whole system. Because they can tailor educational spending more closely to student quality, university systems serving a larger population produce more output per student, holding constant total resources per student. Improved tailoring raises aggregate social welfare, but it does not benefit all students (i.e., it is not Pareto improving). In particular, the lowest ability students at each school will lose when an additional school is introduced into the system, because they will be dropped into a lower tier. Larger systems with more schools will provide a college education to a larger fraction of the population, and they will feature a wealthier and more selective flagship school, at the optimum.

The principal contribution of this paper is that these results, which are broadly consistent with observed stylized facts, can be derived from a very simple model. Most states have a hierarchy of postsecondary institutions exhibiting markedly different levels of resources. The most obvious example is California, where the state's Master Plan for Higher Education clearly lays out a three tier structure comprised of the University of California system, the California State University system, and the California Community Colleges. Even without an explicit plan, most states have a flagship public university, some number of other four year institutions, and a system of local community colleges. Students are distributed among these schools largely on the basis of their measured academic ability, and discontinuous levels of public spending per student in each tier are strongly and positively associated with average ability. On average across the country, instructional expenditure per student in public universities that grant doctorates is more than twice that in community colleges. The difference in total expenditures directly relevant to education is higher.[1] Further, Winston (1999) shows that spending per student and subsidy per student are generally increasing in student quality in the U.S across all universities.

Our paper is closely related to existing work on systems of educational provision at the elementary and secondary level, as well as earlier work on competitive (as distinct from planned) systems of higher education. All of the earlier work that we are aware of contains explicit consideration of peer effects. Arnott and Rowse (1987) study an elementary and secondary system from a planning perspective similar to ours. They consider the allocation of students to various classes within a school, where students vary in ability and classrooms have a level of resources per student that applies to all students in the class. Their principal finding is that any type of partition is possible, depending on the strength of peer effects.[2]

Peer effects are a central feature of other related work. Rothschild and White (1995) analyze competitive outcomes in higher education with peer effects and demonstrate the potential for efficient private provision. Epple and Romano (1998) construct a model of private and public secondary schools in order to analyze the effects of voucher reforms. Epple, Romano and Sieg (2003) and Epple, Romano and Sieg (2006) consider a model of higher education in which universities compete on quality and university differentiation is driven by exogenous endowment differences.

The driving force in our model is complementarity in production. The notion that complementarity leads to positive assortative matching (tiers, in this case) is hardly new. This is the underlying mechanism in the marriage market model of Becker (1973), for example. In the education literature, Arnott and Rowse (1987), Bénabou (1996), Epple and Romano (1998), Epple et al. (2003) and Epple et al. (2006) all derive at least some results that resemble our tiered

[1] Based on authors' calculations of 2004 data from the Integrated Postsecondary Education Data System.

[2] Effinger and Polborn (1999) work with a model that, on the surface, appears similar to ours. They begin, however, by assuming that there are two different schools and that some students are innately better served at the "lower" school. They then solve an allocation problem, under the assumption that attendance at one school versus the other affects wages in a market with imperfect information.

structure.[3] In addition to assuming complementarity between resources and ability, this earlier work considers peer effects, and, in most cases, outcomes are influenced by the distribution of income.

Our model can be interpreted as a simplified case of much of this earlier work. As is often the case, simplification yields both benefits and costs. We show that it takes only two strong (but not unreasonable) assumptions to generate an optimal system that is broadly consistent with the stylized facts of state higher education systems. By eliminating peer effects, we demonstrate that complementarity, along with fixed costs, is sufficient to make a tiered system optimal. Our simple model also allows us to consider some issues that do not appear in the prior literature. In particular, our results on resource discontinuities, our analysis of the optimal number of universities, and our discussion of the selectivity of schools are new, even as they are latent in earlier work.

We cannot, however, claim to have captured everything that might be important in our simple model. As indicated by previous research, markets, peer effects and associations between income and ability are important in higher education systems. Transportation costs (and therefore spatial considerations) and the political economy of education finance are also surely influential. Abstracting from these concerns enables us to isolate the role of complementarity and to build intuition, but it also eliminates consideration of the ways in which these factors may reinforce or counteract our findings.

We also believe that our emphasis on a planner's perspective is of value. In Epple and Romano (1998), Epple et al. (2003) and Epple et al. (2006), schools compete with each other by maximizing quality. While this is a reasonable approach, we think that for the case of public universities in state systems, it is more natural to consider the planner's problem of maximizing total output across a set of schools. While both approaches lead to similar mathematical results on sorting, we think that framing the problem from a planner's perspective is of heuristic value.

We build up our model incrementally throughout the remainder of the paper. In section 2, we introduce the key elements of the model. Section 3 describes the solution when the optimal number of schools is fixed. Section 4 extends the analysis to consider the optimal number of schools. Section 5 concludes. All proofs are relegated to the appendix.

[3] A strand of the literature that studies continuous optimization problems has touched on the implications of complementarity between student ability and student quality. Fernandéz and Galí (1999) is an example. That paper differs significantly from our analysis by considering a continuum of pre-existing schools of exogenous quality.

2 A Description of the Model

We model a social planner's problem. The planner takes as given the distribution of students, the amount of resources available, the education production function, and the fixed cost of establishing a school. The planner chooses the number of universities, selects which students attend each university, and decides how many resources to give each university.

We model educational output as a function of student ability and resources per student at the student's school. We assume that ability and resources are complementary. Our assumptions about the distribution of students will be innocuous, but our assumptions about the curvature of the education production function are key to our results.

We believe a planner's problem is attractive both because it is relatively simple and because it is a good approximation to the real world, where the vast majority of students attend public institutions. In our model, students are not explicitly decision-makers. If they were, however, they would all have unanimous preferences to attend schools with higher resources per student, because this is the only dimension along which universities vary. The planner can therefore use selective admissions to produce the desired allocation. In other words, a planner with control over admissions policy can satisfy all incentive constraints.

We consider a utilitarian social welfare function, so that the social planner seeks to maximize the aggregate level of educational output. Distributional considerations could be modeled by giving the social planner a preference for equality of outcomes (e.g., a concave social welfare function), equality of expenditure (e.g., a loss function for school quality disparities), or equality of access (e.g., a loss function for selective admissions). Courant, McPherson and Resch (2006) argue that distributional considerations may affect the design of higher education because higher education may be a useful instrument for smoothing preexisting differences in welfare. We acknowledge that distributional concerns are interesting, but we focus on the utilitarian case to maintain simplicity and because we believe it is a good characterization of higher education (as opposed to primary and secondary education), where selective admissions prevail and there is typically no presumption of education for all.[4]

[4]There is a class of concave social welfare functions that we could employ without changing any qualitative results. This leads to limited additional insight, at the cost of significant additional notation. A sufficient condition for our results to hold is that the transformation $U(h(\cdot))$ be supermodular and concave, where $h(\cdot)$ is the education production function and $U(\cdot)$ is the social welfare function. The conditions on $U(\cdot)$ and $h(\cdot)$ that ensure this have no

We do not consider peer effects. They are not needed to obtain any of our results, and omitting them simplifies the model and makes the mechanics more transparent. Previous research has included both peer effects and expenditures per student as inputs to education (Arnott and Rowse, 1987; Epple and Romano, 1998; Effinger and Polborn, 1999; Epple et al., 2003), clouding the issue of what drives the model. Our results show that complementarity is sufficient for educational sorting.

The education produced by an individual student is denoted by $h(x, r)$, where x is that student's ability and r is the resources per student at the student's school. Ability follows a continuous, differentiable cumulative distribution, $F(x)$, with a probability density function denoted by $f(x)$ and a finite support bounded by $\underline{x}$ and $\overline{x}$.[5]

To establish a school, the planner pays a fixed cost, θ, and then purchases the variable input into education. We assume that all students at a school receive the same resources per student. In effect, educational resources at the school are a congestible public good. For any level of total resources that a school provides, the level of resources per student depends only on the number of students in the school.

We assume that the education production function, $h(x, r)$, is continuous, twice differentiable, and increasing and concave in each argument. Logically, output should be increasing in ability and resources. We also suppose that it is concave in each element. As a normalization, we assume that students with zero resources produce zero output, and we restrict the domain of h to weakly positive values of resources. Finally, we assume that the education output function exhibits complementarity. This may also be called supermodularity, and it is equivalent to a positive cross partial derivative.

$$
\begin{gathered}
h = h(x, r) \\
\begin{array}{cc}
h_1 > 0 & h_{11} < 0 \\
h_2 > 0 & h_{22} < 0 \\
h_{12} > 0 & h(x, 0) = 0
\end{array}
\end{gathered}
$$

Only complementarity should be a controversial assumption. Complementarity means that, at any given level of resources per student, higher ability students produce more when given a marginal increase in resources. While it is not obvious that this is true in all cases, we find it to be a plausible assumption. We note also that it is pervasive in the literature (Arnott and Rowse, 1987; Epple et al., 2003).

obvious economic interpretation.

[5]A finite support is not necessary generally, but it will be required when we later assume a uniform distribution.

The planner will choose to set up K universities, indexed by $k = 1, \ldots, K$. The planner must pay θK in fixed costs from the total available resources T. What remains, R, the resources net of the fixed costs, is partitioned among the schools. We denote the proportion of R allocated to school k as ρ_k. The planner must also partition the distribution $F(x)$ between schools. For each value of x, the planner allocates a proportion of the distribution to each school, denoted by $p_k(x)$. The total measure of students is denoted by S. The measure of students at a school is denoted by s_k and is equal to $S \int p_k(x) f(x) dx$. Thus, the resources per student at a school, r_k, may be written as $\frac{\rho_k R}{s_k}$.

The planner simultaneously chooses the number of universities and the partition of students and resources. It is useful, however, to write the planner's problem when the number of schools is fixed as a sub-problem. We denote the global value function as V, and the value function when K is fixed as W:

$$
\begin{aligned}
V(T, S, \theta) &= \max_K W(T, S, \theta, K), \text{ where} \\
W(T, S, \theta, K) &= \max_{\{\rho_\mathbf{k}\},\{\mathbf{p_k(x)}\}} S \int_{\underline{x}}^{\overline{x}} h\left(x, \frac{\rho_1 R}{s_1}\right) p_1(x) f(x) dx + \\
& \quad ... + S \int_{\underline{x}}^{\overline{x}} h\left(x, \frac{\rho_K R}{s_K}\right) p_K(x) f(x) dx \\
\text{s.t.} \quad & \theta K + R \leq T \\
& \sum_{k=1}^{K} \rho_k \leq 1 \qquad \text{(P1)} \\
& \rho_k \geq 0 \ \forall \ k \\
& \sum_{k=1}^{K} p_k(x) \leq 1 \ \forall \ x \\
& p_k(x) \geq 0 \ \forall \ k, x
\end{aligned}
$$

Each integral of program P1 represents a school. The output of a school is the integral of individual student outputs with resources equal to r_k, integrated over $p_k(x)f(x)$, the distribution of students assigned to the school.

The first and second constraints are the planner's budget constraint. The third disallows "negatively funded" schools. The fourth and fifth restrict the planner's partition, disallowing negative assignments, while permitting the planner to not educate some students.

In choosing the optimal number of schools, the planner balances the burden of paying the additional fixed costs for more schools against the inefficiency of sending very different types of students to the same school. The planner allocates students and resources, which implicitly sets the resources per student at each school.

First order conditions for this problem can, at least in principle, be established using variational methods. In the interest of clarity, we shall instead demonstrate that the problem can be reduced to a more tractable form.

3 The Optimal Allocation When the Number of Universities is Fixed

We begin by isolating the allocation decision, taking the number of schools as fixed. With a fixed number of universities, the planner's solution is a mapping from the set of students and resources into universities. One class of partitions of the type space involves grouping the highest ability types together in one school, then grouping the next highest ability types in a second school, and so on. We call this a monotonic partition.

Definition 1. *A partition is monotonic if and only if, for least and greatest elements $\underline{x}_k$ and $\overline{x}_k$ in each school, a student x is assigned to school k if and only if $\underline{x}_k \leq x \leq \overline{x}_k$.*[6]

Any partition that results in one school having both higher and lower ability students than another school cannot be monotonic. Any partition that puts two students of the same type into different schools cannot be monotonic. Supermodularity (complementarity) of the underlying education production function is a sufficient condition to make the optimal partition monotonic.

Proposition 1. *If $h(x, r)$ is complementary (supermodular), then the optimal partition of students is monotonic.*[7]

Supermodularity is sufficient to generate educational sorting, even when there are no peer effects. Imagine, instead, a non-monotonic partition between two schools. The allocation can be improved by replacing a lower ability student with a higher ability student in the school with more resources per student.

Corollary 1. *In any optimal monotonic partition, any school that has higher ability students than another school will also have higher resources per student.*

[6] Alternatively, this could be stated as, for least and greatest elements $\underline{x}_k$ and $\overline{x}_k$ in each school, $p_k(x) = 0$ if $x < \underline{x}_k$ or $x > \overline{x}_k$ and $p_k(x) = 1$ if $\underline{x}_k < x < \overline{x}_k$.

[7] All proofs are in the appendix.

Resources and ability are complements. This immediately leads to the conclusion that universities with higher ability students should have more resources per student.

Proposition 1 tells us the shape of the optimal solution, allowing us to rewrite program P1. The planner sets an admissions policy by determining the lowest ability type admitted to each school.[8] We denote the highest type assigned to school k by a_k, with a_0 denoting the lowest type at the lowest school.[9]

$$
\begin{aligned}
V(T,S,\theta) &= \max_K W(T,S,\theta,K), \text{ where} \\
W(T,S,\theta,K) &= \max_{\{\rho_\mathbf{k},\mathbf{a_k}\}} S\int_{a_0}^{a_1} h\left(x,\frac{\rho_1 R}{s_1}\right) f(x)dx + \\
&\quad ... + S\int_{a_{K-1}}^{a_K} h\left(x,\frac{\rho_K R}{s_K}\right) f(x)dx \\
\text{s.t.} \quad & \theta K + R \leq T \\
& 0 \leq a_0 \leq ... \leq a_k \leq a_{k+1} \leq ... \leq a_K \leq 1 \\
& \sum_{k=1}^{K} \rho_k \leq 1 \\
& \rho_k \geq 0 \;\forall\; k
\end{aligned}
\tag{P2}
$$

Program P2 has one fewer constraint than program P1, and the suboptimization problem for W is a standard static optimization problem. One can easily construct a Lagrangean and characterize the first-order conditions for any given K. In principle, the planner can find the optimal allocation for each value of K that is feasible, then choose the best among these.

We find significant heuristic value in further simplifying the problem. First, we assume that the distribution of student ability is uniform on $[0,1]$. This simplifies notation, but does not substantively affect any interpretations. Second, we normalize S to 1. Third, for the remainder of this section only, we assume that the number of schools is fixed at two. Again, this substantially clarifies the tension in the model, and all of the following results are easily translatable to other values of K.

Under these additional assumptions, the planner chooses one value of ρ, the proportion of resources to be allocated to the lower school, and two cut-off

[8] There will be no gaps between the lowest type in one school and the highest type in the next school; otherwise total output could be increased by giving a higher ability student the place of a student at the lower school.

[9] I.e., $p_k(x) = 1$ for $x \in [a_{k-1}, a_k]$ and $p_k(x) = 0$ otherwise.

conditions, the lowest ability type admitted to the lower school, a, and the lowest ability type admitted to the higher school, b.

$$\begin{aligned} \max_{a,b,\rho}\ H(a,b,\rho) = \int_a^b h\left(x, \frac{\rho R}{b-a}\right) dx + \int_b^1 h\left(x, \frac{(1-\rho)R}{1-b}\right) dx \\ \text{s.t.}\quad 0 \le a \le b \le 1 \\ 0 \le \rho \le 1 \end{aligned} \tag{P3}$$

First-order necessary conditions for an interior solution follow from the unconstrained optimization problem. At an interior optimum, the Lagrange multipliers on the inequality constraints are all zero. Only one constraint, $a = 0$, can ever bind at the optimum.[10]

$$H_\rho = \frac{R}{b-a}\int_a^b h_2\left(x, \frac{\rho R}{b-a}\right) dx - \frac{R}{1-b}\int_b^1 h_2\left(x, \frac{(1-\rho)R}{1-b}\right) dx = 0 \tag{1}$$

$$H_a = -h\left(a, \frac{\rho R}{b-a}\right) + \frac{\rho R}{(b-a)^2}\int_a^b h_2\left(x, \frac{\rho R}{b-a}\right) dx = 0 \tag{2}$$

$$\begin{aligned} H_b = & h\left(b, \frac{\rho R}{b-a}\right) - h\left(b, \frac{(1-\rho)R}{1-b}\right) - \frac{\rho R}{(b-a)^2}\int_a^b h_2\left(x, \frac{\rho R}{b-a}\right) dx \\ & + \frac{(1-\rho)R}{(1-b)^2}\int_b^1 h_2\left(x, \frac{(1-\rho)R}{1-b}\right) dx = 0 \end{aligned} \tag{3}$$

Another way to write these first-order conditions is to substitute s_k and r_k back into the equations.

$$H_\rho = \frac{R}{s_1}\int_a^b h_2(x, r_1)\, dx - \frac{R}{s_2}\int_b^1 h_2(x, r_2)\, dx = 0 \tag{1b}$$

$$H_a = h(a, r_1) - \frac{r_1}{s_1}\int_a^b h_2(x, r_1)\, dx = 0 \tag{2b}$$

$$H_b = -h(b, r_1) + h(b, r_2) + \frac{r_1}{s_1}\int_a^b h_2(x, r_1)\, dx - \frac{r_2}{s_2}\int_b^1 h_2(x, r_2)\, dx = 0 \tag{3b}$$

Equation 1 states that the full marginal output of a dollar spent at either school must be the same in equilibrium. Educational production per student

[10] The other inequality constraints, $a = b$, $b = 1$, $\rho = 0$, and $\rho = 1$ all imply that one university is empty and unused. This cannot be optimal. Whenever the fixed cost has been paid, the optimal allocation uses all available schools to tailor resources per student. In the Cobb-Douglas case, which we explore in detail below, with $\underline{x} = 0$, $a = 0$ will not bind because the lowest student produces zero.

depends not only on the total budget of a school, but also on the number of students over which this budget is spread; the key metric is resources per student. The price of an additional unit of resources per student in a school is equal to the size of the school. Rearranging 1 yields:

$$\frac{\text{Price of } r_2}{\text{Price of } r_1} = \frac{s_2}{s_1} = \frac{1-b}{b-a} = \frac{\int_b^1 h_2\left(x, \frac{1-\rho}{1-b}\right) dx}{\int_a^1 h_2\left(x, \frac{\rho}{b-a}\right) dx} = \frac{\text{Marginal Effect of } r_2}{\text{Marginal Effect of } r_1}$$

Equation 2 describes the condition for the lowest ability person who receives education. The first term represents the contribution to education made by the marginal person when he or she is admitted. The second term represents the reduction in education of those already at the school, due to congestion, when an additional student is added. When the marginal person is added to the school, holding the school's total resources fixed, the level of resources per student falls (at a rate of $\frac{\rho R}{(b-a)^2}$), and this causes a decrease (in the amount of $h_2(x, \rho R/(b-a))$) in production for each student in the school. Thus, at the optimum, the direct contribution of the marginal student just offsets the reduction that student causes by congesting resources.

Equation 3 describes a similar condition for the marginal student between schools. Suppose the decision is made to send the best person from the lower school to the upper school. Their direct contribution rises by the amount $h(b, \frac{(1-\rho)R}{1-b}) - h(b, \frac{\rho R}{b-a})$, as a result of attending a school with higher resources per student (recall from corollary 1 that the higher school will have more resources per student at the optimum). This gain is exactly equal to the net crowding effect. The other two terms in 3 are the combined marginal benefit in the lower school of moving the student and the combined marginal loss to the students at the upper school from increased congestion.

This marginal student faces a discontinuity. He or she would produce discretely more at the upper school. Because of the complementarity between resources and student ability, the students at the better school enjoy more resources per student. The top person in the lower tier is almost exactly the same as the lowest person in the upper tier in terms of ability, but there is a discrete gap in their educational outcomes.

4 The Optimal Number of Universities

The above analysis characterized the optimal allocation, taking the number of schools as fixed. The planner must also choose the optimal number of schools.

This analysis is less straightforward because the problem is discrete.[11] We can develop some intuition by looking at the case where $\theta = 0$, which again allows the use of standard calculus. When there is no fixed cost, the optimal solution is to tailor the resources per student to each ability type, with the resources per student rising in student ability.

Proposition 2. *If there are no fixed costs ($\theta = 0$), the optimal solution features a unique level of funding (a unique school) for each student ability that is funded at a positive level. The optimal amount of resources per student, $r(x)$, is an increasing function with $r'(x) = \frac{-h_{21}(x,r(x))}{h_{22}(x,r(x))} > 0$.*

When there are no fixed costs, the planner tailors education quality specifically for each ability level. The proof of proposition 2 solves a basic control problem. The solution demonstrates that resources will be rising in student ability, and that the rate of this increase will depend on the curvature of h. Greater complementarity increases the slope of resources as ability rises. Greater concavity in the value of resources will dampen the relationship.

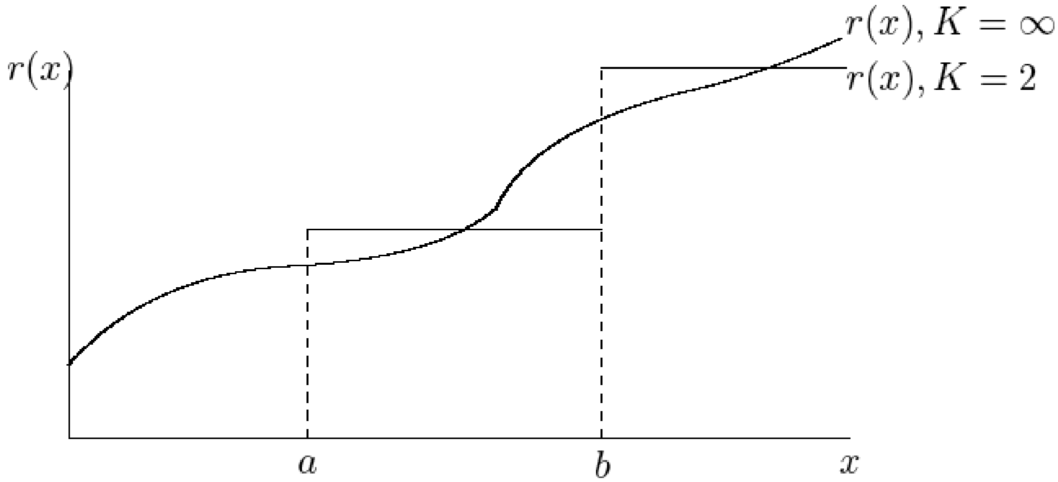
When there are fixed costs, the planner can provide only finite tailoring, which implies that almost all ability types will receive resources different from the infinite school optimum. This creates both winners and losers. Figure 1 shows a hypothetical resources per student function for the no fixed cost case and the same function when $K = 2$. For $K = \infty$, $r(x)$ must be increasing. The area trapped by $r(x)$ will represent the net resources available, R, when the distribution of ability is uniform and measure 1. It is possible, but not necessary, that all students receive some education in this system.

Suppose that $K = 2$ is the constrained optimum (which will be the case for some values of θ). Some measure of students at the bottom may receive no education in the constrained case. Students between a and b attend the lower tier school, and students above b attend the upper tier school. The lowest ability students at each school receive more funding than they would in the case with perfect tailoring. In general, an increase in the ability to tailor resources will increase total educational output, but it will not be Pareto improving. The lowest students at each school will lose if the number of schools increases.

When there are fixed costs, the planner must balance the benefits of tailoring against the costs of setting up new universities. A graphical representation

[11] One may wish to appeal to discrete optimization tools such as integer programming to solve such a problem. Unfortunately, integer programming techniques, such as cutting plane methods, are not applicable because they require first solving the case where variables are not constrained to be integers. This will not work here because the objective function is not defined for non-integer values of K.

Figure 1: Resources per Student as a Function of Student Ability



of the planner's global choice provides further intuition. Figure 2 shows several hypothetical curves in total resources versus total educational production space. These curves are the $W(T, S, \theta, K)$ value functions from program P2, for several values of K, with S and θ held constant. Each curve shows how total output changes as total resources rises, holding fixed the number of schools, and the measure and distribution of students. These curves are increasing and concave. The global optimum, $V(T, S, \theta)$, is the upper envelope of these curves.

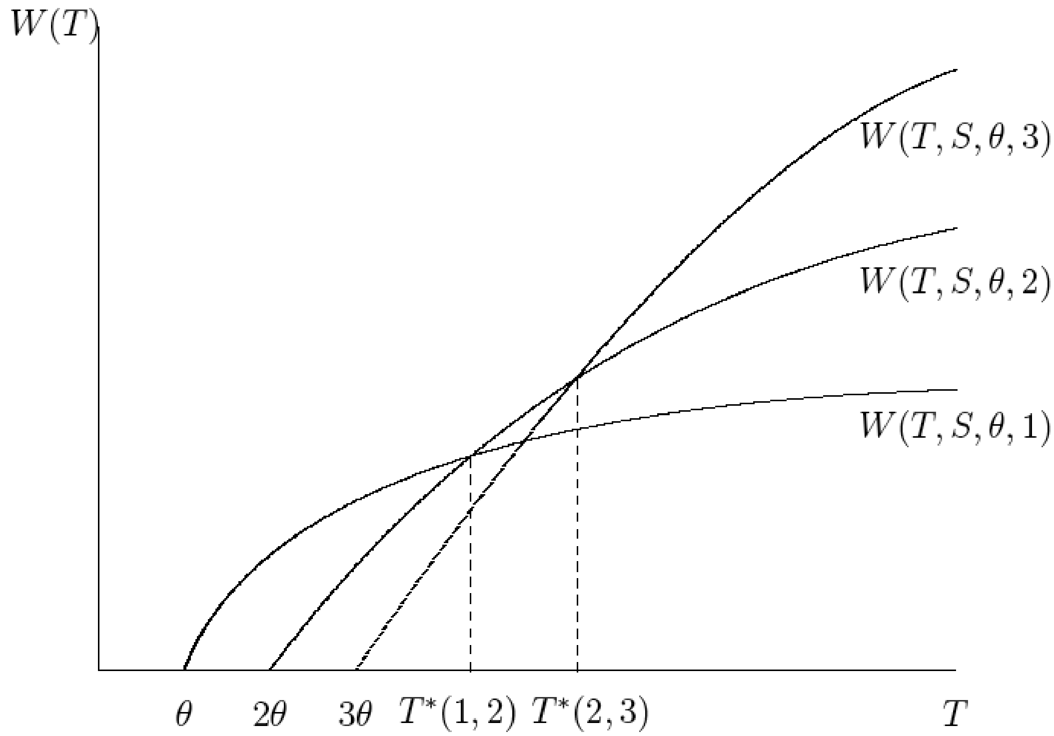
A number of comparative statics can be visualized as the expansion or compression of figure 2. Holding T constant, a fall in θ compresses the graph. Each W curve shifts horizontally to the left, and curves at a higher K shift more. As a consequence, systems with lower θ will be more productive.

Proposition 3. *The average product of resources, $\frac{V}{T}$, of the optimal system is rising in the measure of students when resources per student is held constant and falling in the fixed cost per school.*

The second part of proposition 3 is rather obvious. Average productivity rises when more resources are available for education and fewer are required for paying the fixed cost. The first part follows from the implied economies of scale. Larger systems will spread the fixed cost over more students, allowing more money to be used as an input.

Figure 2 also provides insight into how the optimal number of universities is chosen. For the given value of T, the planner will choose the highest curve.

Figure 2: Total Output versus Total Resources for Several Values of K



Each W curve begins on the T-axis at $T = \theta K$. For T above that point, W is increasing and concave. If each W satisfies the single-crossing property, then the optimal number of schools must be rising in T. Currently, we are unable to prove (or disprove) that the single-crossing property is satisfied without any additional assumptions, though our intuition is that the property will hold for a fairly broad class of functions. This property holds in the Cobb-Douglas case, and we can prove several further results with this functional form.[12]

Proposition 4. *If $h(x,r) = x^{\alpha} r^{\beta}$, with $0 < \alpha, \beta < 1$, then the optimal number of schools, K^*, is weakly rising in total resources, T.*

Proposition 4 implies that richer systems should have more schools, thereby achieving better tailoring. Note, again, that better tailoring does not mean that all students benefit. Within each school, there are students who receive

[12] The assumption of Cobb-Douglas can be slightly relaxed to an assumption that $h(x,r) = \alpha r^{\beta} g(x)$ without changing the proof used. We suspect that this property is true for a broader set of h functions, but the current proof uses the multiplicative separability of Cobb-Douglas, which is a relatively strong assumption.

more resources than they would if perfect tailoring were feasible. Thus, there will be losers from an increase in total system resources if the addition of resources causes a rise in the number of schools. In particular, some portion of the lowest ability students at any given school will experience a decrease in educational quality when K rises. Increases in T are not, therefore, necessarily Pareto-improving, even if resources are dropped exogenously into the system.

Proposition 4 is closely tied to two additional comparative static results, which relate the optimal number of schools to the fixed cost of establishing a school and to the size of a system.

Proposition 5. *If $h(x,r) = x^\alpha r^\beta$, then the optimal number of schools, K^*, is rising in the measure of students when resources per student is held constant and falling in the fixed cost per school.*

The intuition behind proposition 5 is clear from figure 2. A reduction in θ shifts all the curves to the left. Curves with higher K values shift more. Thus, the diagram is contracted, and the cut-off points all move to the left. Holding T constant, K^* must weakly rise as a result. Raising the measure of students, while keeping total resources per student constant, has the same effect on the cut-off points.

University systems that serve a larger population, therefore, should be superior in several ways. Even if they are not richer per student, they should have more universities. They should do a better job of tailoring educational quality to students, and they should produce more per dollar of resources and more per student.

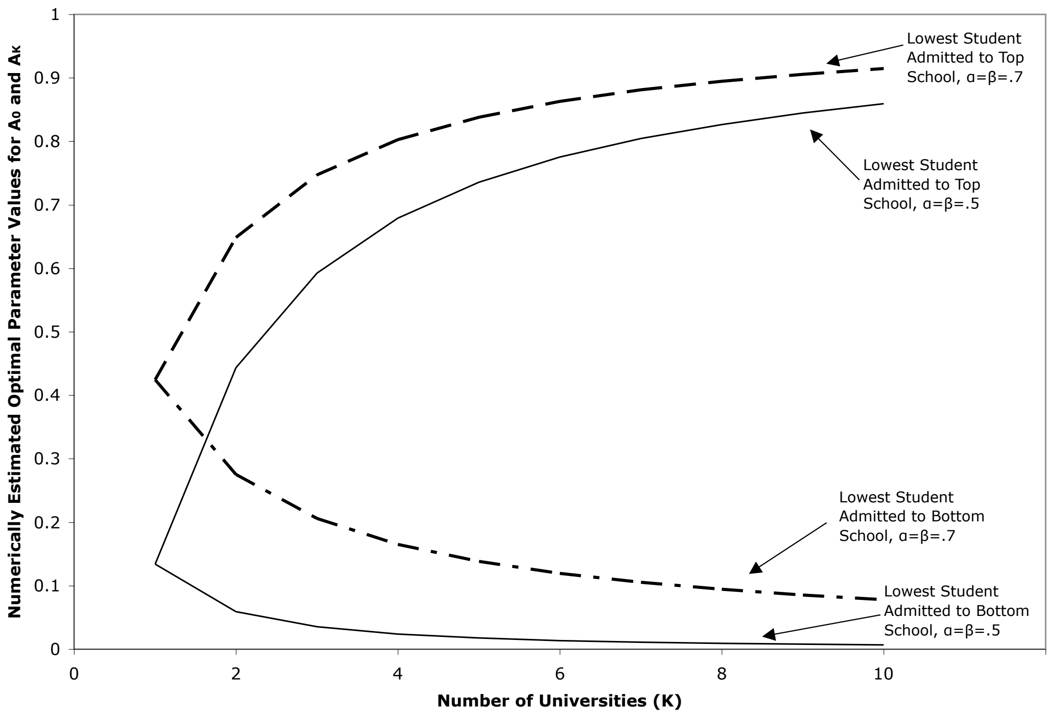
Larger university systems will also serve a greater fraction of the ability distribution, and they will feature more selective flagship universities.

Conjecture 1. *If $h(x,r) = x^\alpha r^\beta$, then the selectivity of the top university will be rising in K, holding R constant.*

This remains a conjecture, because we have been unable to prove this for the general case, but there are reasons to believe that the claim is true. First, it is clearly true in the limit. As the number of schools approaches infinity, the top school will become arbitrarily selective. Second, we have investigated this claim numerically, assuming that ability is distributed uniformly on $[0,1]$. Our assumptions require that $0 < \alpha < 1$ and $0 < \beta < 1$. We performed a grid search over these intervals with a .1 width, for $K = 1$ through $K = 5$.[13] For

[13] The numerically estimated solution is very sensitive to starting values when the parameters are near 0 or 1, which necessitates an extra layer of search. We checked many values close to 0 and 1, and we performed a finer grid search over the middle of the parameter space (from .2 to .8) where starting value sensitivity is reduced.

Figure 3: Numerically Estimated Optimal Values of A_0 and A_K for Selected α and β

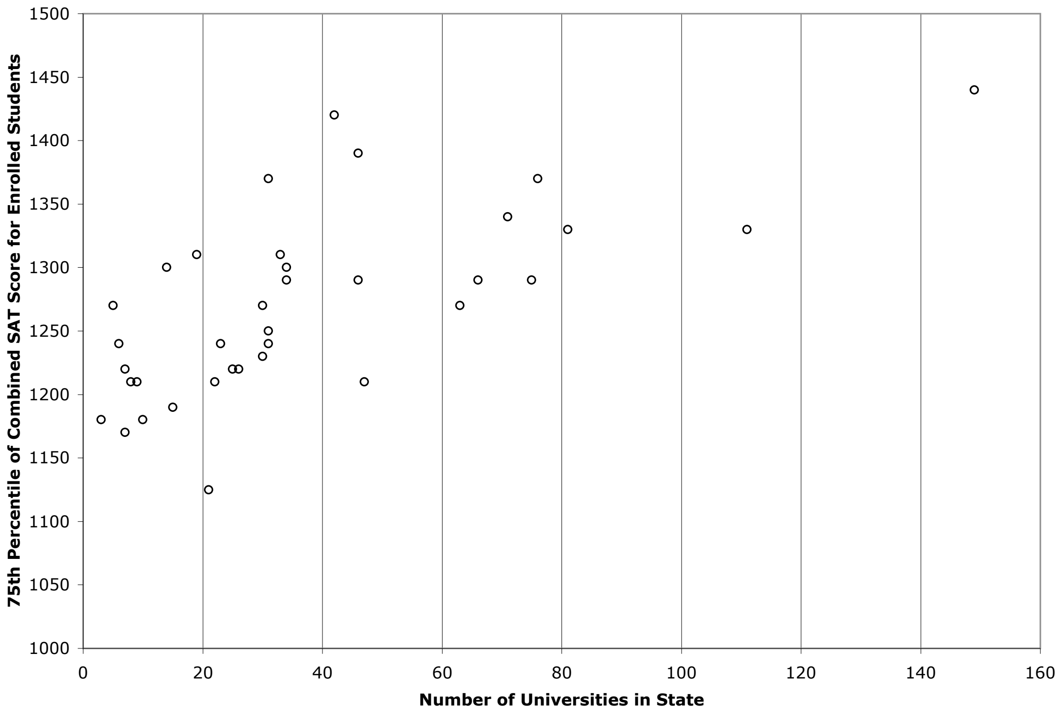


each α, β pair we numerically located the optimal cut points for each value of K and checked that the selectivity of the top school is rising in K. This procedure revealed no counterexamples. These examples, of course, do not prove the conjecture. Note, however, that even if there is some set of values for α, β and K that generate a counterexample to the claim, the predicted relationship will likely still emerge in the real world. A similar result about the low end of the distribution holds in simulations. Systems with more universities will serve a larger fraction of the distribution.[14]

We selected additional parameter values and extended the search up to $K = 10$. Two examples are provided in figure 3 for illustration. When $K = 1$, the lowest type admitted to the top school is the same as the lowest type admitted to the bottom school. As K rises, the lowest type admitted to the top school

[14]An alternative approach is to calibrate the model. We prefer the grid search primarily because we do not believe there is a reliable way to calibrate α and β. Since we find no contradictions to our claim throughout the entire parameter space, we feel that the grid search is more comprehensive than a calibrated example, which would focus on a single pair of α and β.

Figure 4: Flagship SAT Scores versus Number of Universities in State



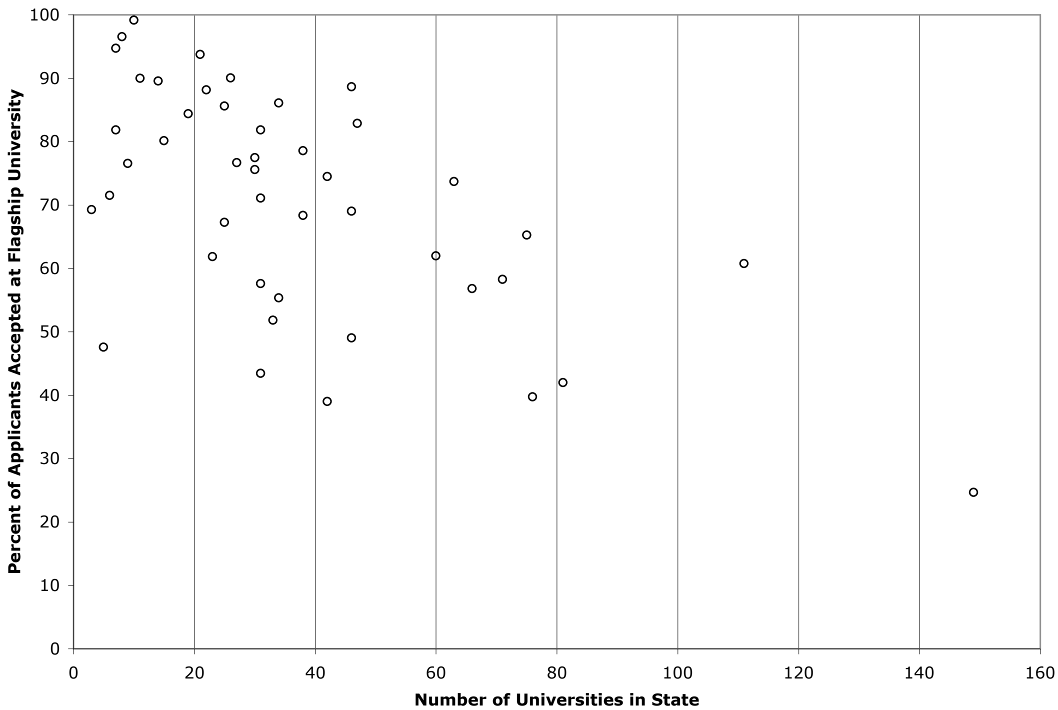
Data include all 36 states with available SAT scores from the 2001 Integrated Postsecondary Education Data System. A regression of 75th Percentile SAT scores at the state flagship on the number of colleges in a state yields the following estimates: SAT = 1213 (14) + 1.52 (.29) * Number of Schools + error, where standard errors are in parentheses.

(A_K) also rises. The limiting argument suggests that as $K \to \infty, A_K \to 1$, giving a sense of how these curves would project forward. A corresponding shape exists for the lowest student admitted to the bottom school, with this value approaching 0 in the limit. The corresponding curves have a similar shape for each of the large number of parameter value pairs that we have examined.

These results suggest that schools in states with a larger number of universities should have more selective flagships. Descriptive data from the Integrated Postsecondary Education Data System on university characteristics in 2001 support this hypothesis.[15] Figure 4 plots the 75th percentile of the combined SAT scores for students at each state's flagship university against the number of two- and four-year public universities in that state. It is clear from the graph that states which have more institutions (better tailoring) feature a more selective flagship university. Figure 5 plots the percentage of applicants

[15]These data are available at http://nces.ed.gov/ipeds/.

Figure 5: Flagship Admissions Acceptance versus Number of Universities in State



Data include all 45 states with available admissions data from the 2001 Integrated Postsecondary Education Data System. A regression of fraction admitted to the flagship on the number of colleges in a state yields the following estimates: Fraction Admitted = 107 (15) * -1.00 (.20) Number of Schools + error, where standard errors are in parentheses.

admitted by the flagship university against the number of public universities in that state. The data again suggest that larger university systems have more selective flagships.

5 Extensions and Conclusions

The purpose of this paper is to provide a framework for analyzing the optimal allocation of students and resources within a system of higher education. Our hope is that future research will enrich the model and test its implications.

Our model does not include tuition.[16] At the optimum, the social cost of moving a student from their assigned university to a better one is the change

[16]The existing literature primarily considers tuition policies that enable ability screening for schools maximizing quality (e.g., Epple and Romano (1998); Epple et al. (2003, 2006)).

in their educational output minus the net crowding effect. If individuals experience a private gain from educational output, there will be some measure of students at any university for whom the private gain from a university upgrade will outweigh the total social cost. This suggests that there are gains to be made by allowing students to pay for an upgrade.

Similarly, students might be willing to pay a premium to attend a university out of state. If the social planner's objective function includes only the education of in-state residents, the optimal tuition policy will be to admit out-of-state students as long as their tuition, at the margin, exceeds the current resources per student at a school. In general, the introduction of tuition policy will make the total amount of system resources an endogenous variable.

Our model also makes empirical predictions about the relationship between the number of universities in a system and the selectivity of its flagship university and about the effects of introducing an additional university to a system. When new universities are introduced, our model suggests that some types of students will experience a reduction in educational quality, while others will experience an increase. At the same time, overall educational output, and the marginal value of additional revenue, should rise with the introduction of a new university. Our hope is that future research will utilize variation in the fixed cost (e.g., land grants, changes in federal support) and the size of the population (e.g., migration, the Baby-Boom) to test and further refine our findings.

In this paper, we focused on a deliberately simple model. Nevertheless, it captures a number of key features about the provision of public higher education. In particular, our model offers a normative explanation for a tiered university system, within which higher ability students receive more resources. It highlights the tradeoff inherent in tailoring education quality to student ability. It also provides a model for understanding the optimal number of universities in a system, and makes suggestions about how university systems should vary.

6 Appendix

Proposition 1. *If $h(x, r)$ is complementary (supermodular), then the optimal partition of students is monotonic.*

Proof: Fix the number of schools and the resources in each school.[17] Suppose

[17] Clearly, if any two schools provide the same resources per student, there would be economies of scale gains to merging the schools. Thus, we can proceed as if the resources per student differs at each school.

the optimal partition is not monotonic. Call the two schools that violate monotonicity 1 and 2, and, without loss of generality, assume 1 has the higher resources per student. If monotonicity fails, then $\exists\ y \in 1 < z \in 2$. The proposed solution produces $h(y, r_1) + h(z, r_2)$. Switching the two students yields $h(y, r_2) + h(z, r_1)$. And,

$$h(y, r_2) + h(z, r_1) > h(y, r_1) + h(z, r_2) \Leftrightarrow$$

$$h(z, r_1) - h(z, r_2) > h(y, r_1) - h(y, r_2)$$

which is a definition of supermodularity, since $z > y$ and $r_1 > r_2$. **QED.**

Corollary 1. *In any optimal monotonic partition, any school that has higher ability students than another school will also have higher resources per student.*

Proof: Suppose that the optimal partition is monotonic, with students in 1 being higher ability than students in 2, but with $r_2 > r_1$. By supermodularity, swapping any two students between schools raises output. **QED.**

Proposition 2. *If there are no fixed costs ($\theta = 0$), the optimal solution features a unique level of funding (a unique school) for each student ability that is funded at a positive level. The optimal amount of resources per student, $r(x)$, is an increasing function with $r'(x) = \frac{-h_{21}(x,r(x))}{h_{22}(x,r(x))} > 0$.*

Proof: Part I: Suppose that there is a school with positive resources and two or more distinct ability types. Then there exists some school with both y and z with $z > y$. Let s denote the total measure of students at the school. Without loss of generality, suppose that the measure of each type is the same, $s/2$. Then, the output of the proposed optimum can be written:

$$V = \frac{1}{s} h\left(y, \frac{\rho R - \varepsilon}{s}\right) + \frac{1}{s} h\left(z, \frac{\rho R + \varepsilon}{s}\right) \tag{4}$$

where $\varepsilon = 0$ and total funding at the school is ρR. We show that it is optimal to set $\varepsilon > 0$, which is equivalent to separating y and z into two different schools:

$$\frac{\partial V}{\partial \varepsilon} = -h_2\left(y, \frac{\rho R - \varepsilon}{s}\right) + h_2\left(z, \frac{\rho R + \varepsilon}{s}\right) > 0$$

The last inequality follows directly from supermodularity and contradicts the optimality of the proposed solution.

Part II: The infinite school problem may be written as a control problem:

$$\max_{r(x)} \int_0^1 h(x, r(x))dx$$
$$\text{s.t.} \quad \int_0^1 r(x)dx = T$$

It can easily be shown that the Hamiltonian leads to a degenerate solution with $h_2(x, r(x)) = \beta \in \mathbb{R}$. This is an implicit function, and the conditions of the implicit function theorem are satisfied because $h_{22}(\cdot) \neq 0$. The implicit function theorem yields the final result, which we sign from our assumption of concavity and complementarity:

$$r'(x) = \frac{-h_{21}(x, r(x))}{h_{22}(x, r(x))} > 0.$$

QED.

Proposition 3. *The average product of resources, $\frac{V}{T}$, of the optimal system is rising in the measure of students when resources per student is held constant and falling in the fixed cost per school.*

Proof: Fix T and S. Lowering θ relaxes the resource constraint for any value of K. $V(T, S, \theta)$ must therefore rise. Since T is fixed, $\frac{V}{T}$ must rise.

A rise in the measure of students when resources per student is held constant means that S rises but $\frac{T}{S}$ is fixed. We can write this as a γ proportional change, with $\gamma > 1$. Denote the value function as $W(T, S, \theta, K)$. For any value of K, output per dollar of total resources can be written:

$$\begin{aligned}
\frac{W(\gamma T, \gamma S, \theta, K)}{\gamma T} &= \max_{\rho_k, a} \frac{\gamma S}{\gamma T} \sum_{k=1}^{K} \int_{a_k - 1}^{a_k} h\left(x, \frac{\rho_k(\gamma T - \theta K)}{\gamma s_k}\right) f(x)dx \\
&= \max_{\rho_k, a} \frac{S}{T} \sum_{k=1}^{K} \int_{a_k - 1}^{a_k} h\left(x, \frac{\rho_k(\gamma T - \theta K)}{\gamma s_k}\right) f(x)dx \\
&> \max_{\rho_k, a} \frac{S}{T} \sum_{k=1}^{K} \int_{a_k - 1}^{a_k} h\left(x, \frac{\rho_k(T - \theta K)}{s_k}\right) f(x)dx \\
&= \frac{W(T, S, \theta, K)}{T}
\end{aligned}$$

The second equality uses the envelope theorem, which tells us that the optimal cut-points will not change when the parameters are varied in small amounts. The inequality uses the fact that $\frac{\partial}{\partial \gamma} \frac{\rho_k(\gamma T - \theta K)}{\gamma s_k} = \frac{\rho_k \theta K}{\gamma s_k} > 0$. Since this is true of any K, it must be true for the optimal K. **QED.**

Proposition 4. *If* $h(x,r) = x^{\alpha} r^{\beta}$, *with* $0 < \alpha, \beta < 1$, *then the optimal number of schools,* K^*, *is weakly rising in total resources,* T.

Proof: Define $W(T, S, \theta, K)$ to be the constrained solution, when the number of universities is fixed at K, and denote the derivative of $W(\cdot)$ with respect to T by W_T. In the Cobb-Douglas case, we can relate W and W_T:

$$\begin{aligned}
W(T,S,\theta,K) &= \max_{\rho_k,a} S \sum_{k=1}^{K} \int_{a_{k-1}}^{a_k} x^{\alpha} \left(\frac{\rho_k R}{(a_k - a_{k-1})S} \right)^{\beta} dx \\
&= \max_{\rho_k,a} S \left(\frac{R}{S} \right)^{\beta} \sum_{k=1}^{K} \int_{a_{k-1}}^{a_k} x^{\alpha} \left(\frac{\rho_k}{a_k - a_{k-1}} \right)^{\beta} dx \\
&= \max_{\rho_k,a} S^{1-\beta} (T - \theta K)^{\beta} \sum_{k=1}^{K} \int_{a_{k-1}}^{a_k} x^{\alpha} \left(\frac{\rho_k}{a_k - a_{k-1}} \right)^{\beta} dx \\
W_T(T,S,\theta,K) &= \max_{\rho_k,a} \beta S^{1-\beta} (T - \theta K)^{\beta-1} \sum_{k=1}^{K} \int_{a_{k-1}}^{a_k} x^{\alpha} \left(\frac{\rho_k}{a_k - a_{k-1}} \right)^{\beta} dx
\end{aligned}$$

The vector of ρ and a values that maximize the objective function will also maximize the marginal value of resources, W_T, since W_T is an affine transformation of W: $W_T(T,S,\theta,K) = \frac{\beta}{T-\theta K} W(T,S,\theta,K)$. Now, consider any two numbers of universities, with $K > \hat{K}$:

$$\begin{aligned}
W_T(T,S,\theta,K) &= \frac{\beta}{T-\theta K} W(T,S,\theta,K) \\
&> \frac{\beta}{T-\theta K} W(T,S,\theta,\hat{K}) \\
&> \frac{\beta}{T-\theta \hat{K}} W(T,S,\theta,\hat{K}) \\
&= W_T(T,S,\theta,\hat{K})
\end{aligned}$$

Since the derivative of W with respect to T is higher the higher is K, the family of W functions will satisfy the single crossing property in the $T-W$ plane. For each $K > \hat{K}$ and $T > \hat{T}$, $W(\hat{T},S,\theta,K) > W(\hat{T},S,\theta,\hat{K}) \Rightarrow W(T,S,\theta,K) > W(T,S,\theta,\hat{K})$. As is illustrated in figure 2, W will be zero up until $T = \theta K$. So, W functions with higher K values start rising at a later point.

$V(T,S,\theta)$, the optimum when K is a choice variable, is the upper envelope of the family of W functions in figure 2. Because of the single crossing property, this upper envelope must lie on a W for a weakly higher K as T rises. Thus, K^* is rising in T. **QED.**

Proposition 5. *If $h(x,r) = x^\alpha r^\beta$, then the optimal number of schools, K^*, is rising in the measure of students when resources per student is held constant and falling in the fixed cost per school.*

Proof: Define $T^*(i,j)$ as the T that solves $W(T^*(i,j), S, \theta, i) = W(T^*(i,j), S, \theta, j)$, as in figure 2. Define $Q(K) = \max_{\rho_k, a} \sum_{k=1}^{K} \int_{a_{k-1}}^{a_k} x^\alpha \left(\frac{\rho_k}{a_k - a_{k-1}}\right)^\beta dx$. For the Cobb-Douglas case, as is shown in the proof of proposition 4, $W(T, S, \theta, K) = S^{1-\beta}(T - \theta K)^\beta Q(K)$. Therefore, $T^*(i,j)$, which will be unique if it exists by the single-crossing property from proposition 4, solves

$$S^{1-\beta}(T^*(i,j) - \theta i)^\beta Q(i) = S^{1-\beta}(T^*(i,j) - \theta j)^\beta Q(j) \tag{5}$$

Without loss of generality, assume $i < j$. Totally differentiate equation 5 with respect to T and θ:

$$\begin{aligned}&(dT^*(i,j) - id\theta)\beta S^{1-\beta}(T^*(i,j) - \theta i)^{\beta-1} Q(i)\\ &= (dT^*(i,j) - jd\theta) S^{1-\beta}(T^*(i,j) - \theta j)^\beta Q(j)\end{aligned}$$

Rearrange:

$$\begin{aligned}\frac{dT^*(i,j)}{d\theta} &= \frac{j\beta S^{1-\beta}(T^*(i,j) - \theta j)^\beta Q(j) - i\beta S^{1-\beta}(T^*(i,j) - \theta i)^{\beta-1} Q(i)}{\beta S^{1-\beta}(T^*(i,j) - \theta j)^\beta Q(j) - \beta S^{1-\beta}(T^*(i,j) - \theta i)^{\beta-1} Q(i)}\\ &= \frac{jW_T(T^*(i,j), S, \theta, j) - iW_T(T^*(i,j), S, \theta, i)}{W_T(T^*(i,j), S, \theta, j) - W_T(T^*(i,j), S, \theta, i)}\\ &> 0\end{aligned}$$

The last two steps follow directly from the analysis in the proof of proposition 4. This shows that all cut-off points rise when θ rises. This implies that, when T is held constant, a rise in θ must weakly decrease the number of cut-off points passed with total resources T, which is equivalent to a weakly falling K^*.

For the size result, note that $W(\gamma T, \gamma S, \theta, K) = (\gamma S)^{1-\beta}(\gamma T - \gamma \frac{\theta}{\gamma} K)^\beta Q(K) = \gamma W(T, S, \frac{\theta}{\gamma}, K)$. Since γ does not depend on any of the parameters, the cut-off points for the $\gamma W(T, S, \frac{\theta}{\gamma}, K)$ system are equivalent to the cut-off points for the $W(T, S, \frac{\theta}{\gamma}, K)$ system. Thus, an increase in the measure of students, holding constant total resources per student, which is equivalent to choosing $\gamma > 1$, is equivalent in its effect on K^* to a reduction of θ to $\frac{\theta}{\gamma}$. Since K^* was proved above to be weakly falling in θ, it must be weakly rising in the measure of students. **QED.**

References

Arnott, Richard and John Rowse, "Peer Group Effects and Educational Attainment," *Journal of Public Economics*, April 1987, *32* (3), 287–305.

Becker, Gary S., "A Theory of Marriage: Part 1," *Journal of Political Economy*, July - August 1973, *91* (4), 813–846.

Bénabou, Roland, "Equity and Efficiency in Human Capital Investment: The Local Connection," *The Review of Economic Studies*, April 1996, *63* (2), 237–264.

Courant, Paul N., Michael McPherson, and Alexandra M. Resch, "The Public Role in Higher Education," *National Tax Journal*, June 2006, *59* (2), 291–318.

Effinger, Matthias R. and Mattias K. Polborn, "A Model of Vertically Differentiated Education," *Journal of Economics*, February 1999, *69* (1), 53–69.

Epple, Dennis and Richard E. Romano, "Competition between Private and Public Schools, Vouchers, and Peer-Group Effects," *American Economic Review*, March 1998, *88* (1), 33–62.

_ , Richard Romano, and Holger Sieg, "Peer Effects, Financial Aid and Selection of Students into Colleges and Universities: An Empirical Analysis," *Journal of Applied Econometrics*, September - October 2003, *18* (5), 501–525.

_ , _ , and _ , "Admission, Tuition, and Financial Aid Policies in the Market for Higher Education," *Econometrica*, July 2006, *74* (4), 885–928.

Fernandéz, Raquel and Jordi Galí, "To Each According to...? Markets, Tournaments, and the Matching Problem with Borrowing Constraints," *The Review of Economic Studies*, October 1999, *66* (4), 799–824.

NCES, *Digest of Education Statistics 2004*, Government Printing Office, U.S. Department of Education, National Center for Education Statistics, 2005.

_ , *Projections of Education Statistics to 2014*, Government Printing Office, U.S. Department of Education, National Center for Education Statistics, 2005.

Rothschild, Michael and Lawrence J. White, "The Analytics of the Pricing of Higher Education and Other Services in Which the Customers Are Inputs," *Journal of Political Economy*, June 1995, *103* (3), 573–586.

Winston, Gordon C., "Subsidies, Hierarchy and Peers: The Awkward Economics of Higher Education," *Journal of Economic Perspectives*, Winter 1999, *13* (1), 13–36.