Supporting Online Material for "Time perspective and climate change policy," by L. Karp and Y. Tsur

This online document contains proofs and additional numerical analysis of the above-mentioned article. Reference to numbered items (equations, propositions, lemmas, figures) that appear in the article retains the original (article) numbers.

A Proofs

To simplify notation we assume that $g(\eta - 1) = 0$. The proofs extend to the general case by substituting $\tilde{\delta} \equiv \delta + g(\eta - 1)$ and $\tilde{\gamma} \equiv \gamma + g(\eta - 1)$ for δ and γ, respectively.

Lemma 1: *Consider the game in which the payoff at time t equals expression (9); the regulator at time t chooses $w(t) \in \Omega \subset R$, taking as given her successors' control rule $\hat{\chi}(z)$; and the state variables h and y obey equations (4) and (8). Let $V(h)$ equal the value of expression (9) in a MPE (the value function). A MPE control rule $\chi(h) \equiv \hat{\chi}(z)$ satisfies the (generalized) dynamic programming equation (DPE):*

$$K(h) + (\tilde{\gamma} + h)\, V(h) = \max_{w \in \Omega} \{U(w) + q(h, w)V'(h)\} \tag{10}$$

with the "side condition"

$$K(h) \equiv (\delta - \gamma)(1 - \beta) \int_0^\infty e^{-\left(\tilde{\delta}\tau + y(t,\tau)\right)} U\left(\chi(h(t + \tau))\right) d\tau. \tag{11}$$

Proof. The proof is almost direct consequence of Proposition 1 and Remark 2 in [16]. In that paper the state variable is a scalar, but the same results hold (making obvious changes in notation) when the state is a vector, as in the present case. Our state variable is $z \equiv (h, y)$ and the flow of utility (prior to the event) is $e^{-y(t)}U(w(t))$. Specializing equation (5) of [16] to our setting, and using the hyperbolic discount factor in equation (2), yields the generalized DPE

$$\hat{K}(z) + \gamma W(z) = \max_{w\in\Omega}\left(e^{-y(t)}U(w(t)) + W_h g + W_y h\right), \tag{A.1}$$

where $W(z)$ is the value function (with subscripts denoting partial differentiation) and

$$\hat{K}(z) = (\delta-\gamma)(1-\beta)\int_0^\infty e^{-(\delta t+y(t))}U(\hat{\chi}(z))\,dt \tag{A.2}$$

is implied by equation (4) and Remark 2 of [16]

Use the "trial solution" $W(z) = e^{-y}V(h)$ and $\hat{K}(z) = e^{-y}K(h)$, so $W_y = -e^{-y}V(h)$ and $W_h = e^{-y}V'(h)$. Substituting these expressions into equation (A.1), canceling e^{-y} and rearranging, yields equation (10). Conclude that $\hat{\chi}(z) = \chi(h)$: the equilibrium control depends only on the hazard rate.

Conditional on survival up to time t, the probability of survival until time $s > t$ equals $\exp\left(-\int_t^s h(\tau)d\tau\right) = \exp(-y(s)+y(t))$. Use this fact and the trial solution to rewrite equation (A.2) as

$$\begin{aligned} K(h(t)) &= (\delta-\gamma)(1-\beta)\,e^{y(t)}\int_t^\infty e^{-\delta(s-t)}\exp\left(-\int_t^s h(\tau)d\tau\right)e^{-y(t)}U(\chi(h(s)))\,ds \\ &= (\delta-\gamma)(1-\beta)\int_t^\infty e^{-\delta(s-t)}\exp\left(-\int_t^s h(\tau)d\tau\right)U(\chi(h(s)))\,ds \end{aligned} \tag{A.3}$$

Setting $t = 0$ in equation (A.3) produces equation (11). ∎

Lemma 3: *The functions $\pi(h)$ and $\sigma(h)$ are increasing over $(0, a)$ with $\pi(a) = \sigma(a) = 1$, and $\sigma(h)$ is concave.*

Proof. Define

$$\varpi(h) \equiv \pi(h)^{-1} = 1 - \mu(a-h)\xi'(h).$$

Differentiating, using equation (17), we obtain

$$\varpi'(h) = \mu\xi'(h) - \mu(a-h)\xi''(h) < 0.$$

Thus,

$$\pi'(h) = -\varpi'(h)/\varpi(h)^2 > 0.$$

Differentiating (16), using equation (18), gives

$$\sigma'(h) = -\mu\nu'(h) + \mu(a-h)\nu''(h) > 0. \tag{A.4}$$

To establish $\sigma''(h) < 0$, use equation (18) and differentiate three times to obtain $\nu'''(h) < 0$. Differentiating equation (A.4) gives

$$\sigma''(h) = -2\mu\nu''(h) + \mu(a-h)\nu'''(h) < 0.$$

By inspection $\pi(a) = \sigma(a) = 1$. ∎

Proposition 1: *There exists a pure strategy stationary MPE for all* $0 < x < 1$ *and all initial conditions* $h = h_0 \in (0, a)$ *if and only if*

$$\pi(h) < \sigma(h), \ h \in (0, a). \tag{23}$$

Under inequality (23), there exists a MPE with perpetual stabilization ($w \equiv 1$*) if and only if at the initial hazard* h *the cost of stabilization satisfies*

$$x < x^U(h) \equiv 1 - \pi(h); \tag{24}$$

there exists a MPE with perpetual BAU ($w \equiv 0$*) if and only if at the initial hazard* h *the cost of stabilization satisfies*

$$x > x^L(h) \equiv 1 - \sigma(h). \tag{25}$$

Proof. We first establish sufficiency of inequality (23) using a constructive proof, which also establishes the claims associated with inequalities (24) and (25). We then show necessity of inequality (23) using a proof by contradiction.

Sufficiency Suppose that $\sigma > \pi$ for $h \in (0, a)$. We show that there exists a MPE that satisfies $w \equiv 1$ (perpetual stabilization) if and only if the initial condition $h_0 = h$ satisfies equation (24). In a MPE with perpetual stabilization, it is optimal for the current regulator to stabilize given that she believes that future values of h lie in the stabilization region (so she believes that all subsequent regulators will stabilize). The belief that future values of h lie in the stabilization region (a belief we test below) means that for initial conditions in the interior of the stabilization region the value function is given by $V^S(h)$, defined in equation (22), and

$$V^{S\prime}(h) = U(1)\xi'(h)$$

with $\xi'(h)$ obtained using equation (17).

Using equation (10) (and the belief that future values of h lie in the stabilization region), it is optimal for the current regulator to stabilize if and only if

$$U(1) \geq U(0) + \mu(a - h)U(1)\xi'(h)$$

or

$$\frac{U(1)}{U(0)} \geq \pi(h). \tag{A.5}$$

If inequality (A.5) is satisfied with *strict inequality* (as the Proposition requires) at the current time, then regardless of whether the current regulator uses stabilization or BAU, the inequality is satisfied at neighboring times (the near future). Thus, the current regulator's beliefs that future regulators will stabilize are consistent with equilibrium, regardless of the actions taken by the current regulator. If inequality (A.5) is not satisfied, then clearly perpetual stabilization is not an equilibrium. We consider below the case where the weak inequality (A.5) holds with equality.

We turn now to the equilibrium with perpetual BAU. In a MPE with perpetual BAU, it is optimal for the current regulator to follow BAU given that she believes all subsequent regulators will follow BAU. This belief implies that the value function is given by $V^B(h)$, defined in equation (21). It is optimal for the current regulator to pursue BAU if and only if $U(0) +$

$\mu(a-h)U(0)\nu'(h) > U(1)$ or, equivalently, if and only if

$$\frac{U(1)}{U(0)} < \sigma(h) \equiv 1 + \mu(a-h)\nu'(h),$$

establishing condition (25).

To complete the demonstration that perpetual stabilization is an equilibrium, it is necessary to confirm that if equation (25) is satisfied at time t when the hazard is h, then it is also satisfied at all subsequent times, so that the regulator's beliefs are confirmed. The hazard is increasing on the BAU equilibrium path (and non-decreasing on any feasible path), so it is sufficient to show that $\sigma'(h) > 0$. This inequality was established in Lemma 3.

Now we return to the case where inequality (A.5) is satisfied with equality. We want to show that in this case, stabilization is not an equilibrium action. Suppose to the contrary that it is optimal to stabilize when inequality (A.5) is satisfied with equality. From equation (14), the current regulator wants to use BAU if and only if $U(1) < U(0) + \mu(a-h)V'(h)$. In order to evaluate the right side of this inequality, we need to know the value of $V'(h)$; this (shadow) value of course depends on the behavior of future regulators.

Because $\pi'(h) > 0$ from Lemma 3, if the current regulator uses BAU, h increases and the state is driven out of the stabilization region. Therefore, the current regulator can discard the possibility that (if she were to use BAU) all future regulators would stabilize. Future actions could lead to only one of two possible equilibrium trajectories: (i) All future regulators will follow BAU; or (ii) future regulators will follow BAU until the state h reaches a threshold, say $h_0 < \tilde{h} < a$, after which all regulators stabilize. There are no other possibilities, because once the state enters a stabilization region it does not leave it. This fact is a consequence of our restriction to pure strategy equilibria. However, alternative (ii) cannot occur, because $\tilde{h}$ lies to the right of the curve $\pi(h)$, and therefore is not an element of the stabilization region. Thus, the only equilibrium belief for the current regulator is that the use of BAU (and the subsequent increase in h) will cause all future regulators to use BAU. Consequently, where inequality (A.5) is satisfied with equality, it must be the case that $V'(h) = V^{B\prime}(h) = U(0)\nu'(h)$. The assumption that $\sigma(h) > \pi(h)$ implies that $\pi(h)$ lies in the region where perpetual BAU is an equilibrium strategy. Thus, $\pi(h)$ does not lie in the stabilization region, as asserted by the proposition.

Necessity: We use a proof by contradiction, consisting of two parts, to establish necessity. The first part shows that $\sigma(h) < \pi(h)$ cannot hold, and the second part shows that it cannot be the case that $\sigma(h) = \pi(h)$ at any points in $(0, a)$.

For the first part, suppose that for some interval $\sigma(h) < \pi(h)$. Figure A.1 helps to simplify the proof. This figure shows a situation where $\sigma(h) < \pi(h)$ for small h, but it is clear from the following argument that the region over which $\sigma(h) < \pi(h)$ is irrelevant. (An obvious variation of the following argument can be used regardless of the region over which $\sigma < \pi$, because both of these curves are monotonic.) Suppose that the value of $\frac{U(1)}{U(0)}$ lies between the vertical intercepts of the curves, as shown in the figure; e.g. $\frac{U(1)}{U(0)} = d$. Define h_1 implicitly by $\sigma(h_1) = d$. we want to establish that for any initial condition $h_0 = h < h_1$ there are no pure stationary MPE. Perpetual stabilization is not an equilibrium because $d < \pi(h_1)$, and perpetual BAU is not an equilibrium because $d > \sigma(h_1)$. The only remaining possibility is to follow BAU until the hazard reaches a level $\bar{h} < h_1$ and then begin perpetual stabilization. (Recall that once the state enters the stabilization set it cannot leave that set.) However, this trajectory cannot be an equilibrium because the subgame beginning at $\bar{h}$ cannot lead to perpetual stabilization (because the point (h_1, d) lies below the curve π).

For the second part, suppose that $\sigma(h) \geq \pi(h)$ with equality holding at one or more points in $(0, a)$ (that is, the graphs are tangent at one or more points). Let $\hat{h}$ be such a point. The argument above under "sufficiency" establishes that if $\frac{U(1)}{U(0)} = \pi\left(\hat{h}\right)$, then at $h = \hat{h}$ (where equation (A.5) holds with equality) neither perpetual stabilization not perpetual BAU are MPE.

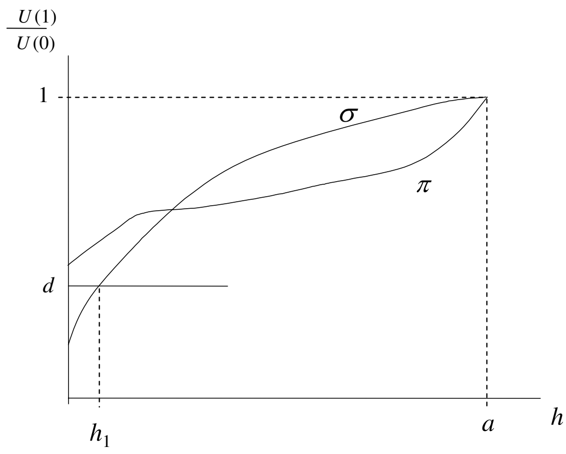

Figure A.1: Graphs of $\sigma(h)$ and $\pi(h)$ that do not satisfy inequality (23).

The only remaining possibility would be to follow BAU for a time and then switch to stabilization in perpetuity. However, that cannot be an equilibrium trajectory, because the initial period of BAU drives the h above $\hat{h}$, where $\frac{U(1)}{U(0)} < \pi(h)$, so the subsequent stabilization period cannot be part of a MPE. Therefore, at $h = \hat{h}$ there is no MPE if $\frac{U(1)}{U(0)} = \pi\left(\hat{h}\right)$. ∎

Proposition 2: *Suppose that Condition (23) is satisfied.* (*i*) *For* $x > 1 - \pi(h)$ *the unique (pure strategy) MPE is perpetual BAU.* (*ii*) *There are no equilibria with "delayed BAU".* (*iii*) *A necessary and sufficient condition for the existence of equilibria with delayed stabilization is*

$$\Theta(h) < x < 1 - \pi(h). \qquad (27)$$

(*iv*) *For all parameters satisfying* $0 \leq h \leq a$, $0 < \beta < 1$, $\delta \neq \gamma$, *and* $\mu > 0$, *a MPE with delayed stabilization exists for some* $x \in (0, 1)$.

Proof. We use the following definition

$$h_\pi(x) \equiv \begin{cases} \pi^{-1}(1-x) & \text{for } x \in [0, 1 - \pi(0)) \\ 0 & \text{for } x \in [1 - \pi(0), 1] \end{cases}$$

Hazard rates that satisfy $h > h_\pi(x)$ lie above the curve $1-\pi$ in Figure 1.

(i) The stabilization set is absorbing, because if a (pure strategy) MPE calls for a regulator to stabilize, the hazard never changes. By Proposition 1, there are no equilibria with perpetual stabilization when $h(0) \geq h_\pi$, and there is an equilibrium with perpetual BAU. The latter is therefore the unique equilibrium. Claim (ii) follows immediately from the fact that the stabilization set is absorbing

(iii) We now consider the case where $h(0) < h_\pi$; equivalently, $x < 1-\pi\,(h)$. From Proposition 1 we know that there is an equilibrium with perpetual stabilization for these initial conditions; and we know that there is an equilibrium with perpetual BAU if x lies between the curves $1-\pi$ and $1-\sigma$. Since the stabilization set is absorbing, we do not need to consider the possibility of equilibria that begin with stabilization and then switch to BAU. Thus, we need only find a necessary and sufficient condition under which there is a "delayed stabilization" equilibrium, i.e. one that begins with BAU and switches to stabilization when the state reaches a threshold $\tilde{h} > h\,(0)$. To conserve notation, throughout the remainder of this proof we use h to denote an initial condition, and use $h\,(\tau)$, with $\tau \geq 0$, to denote a subsequent value of the hazard when regulators use a MPE.

Define two sets, $A = \left\{h \mid h_a \leq h < \tilde{h}\right\}$ and $B = \left\{h \mid \tilde{h} \leq h < h_b\right\}$, where $h_a < \tilde{h} < h_b < h_\pi$. The MPE for initial conditions in set B is to stabilize, and the MPE for initial conditions in set A is to follow BAU. The existence of B follows from the fact that it is an equilibrium to stabilize for any initial conditions in $[0, h_\pi)$ (in view of Proposition 1). In addition, h remains constant when the regulator stabilizes. Therefore, any subset of the interval $[0, h_\pi)$ qualifies as the set B.

The existence of A is not obvious. We cannot rely on the proof of Proposition 1, since that proof applies to the case where the regulator follows BAU in perpetuity. Here we are interested in the case where the regulator switches from BAU to stabilization at a finite time. We obtain the necessary and sufficient condition for the existence of a set A with positive measure.

Suppose (provisionally) that the set A exists. We define the value function for initial conditions in $A \cup B$ as $V(h;\tilde{h})$. We include the second argument in order to emphasize the dependence of the payoff on the switching value $\tilde{h}$. For convenience, we repeat the definition of the value function, given the initial condition $h \in A \cup B$.

$$\begin{aligned} V\left(h;\tilde{h}\right) &= \int_0^\infty e^{-y(\tau)}\theta\left(\tau\right)U(\chi\left(h(\tau)\right)d\tau \;\text{ with } \chi\left(h\right) = \left\{\begin{array}{c} 0 \text{ for } h \in A \\ 1 \text{ for } h \in B \end{array}\right\}, \\ y(\tau) &= \int_0^\tau h(s)ds, \quad h\left(s\right) = \left\{\begin{array}{c} \min\left(a - \left(a-h\right)e^{-\mu s}, \tilde{h}\right) \text{ for } h \in A \\ h \text{ for } h \in B \end{array}\right\}. \end{aligned}$$

Note that for $h\left(\tau\right) \in A$, $h\left(\tau\right)$ is a function of the initial condition, h.

For $h \in A$ the regulator chooses BAU (under the candidate program). Using equation (14), this action is part of an equilibrium if and only if

$$U\left(0\right) - U\left(1\right) > -\mu\left(a-h\right)V_h(h;\tilde{h}). \tag{A.6}$$

In order to determine when this inequality holds, we need to evaluate $V_h(h;\tilde{h})$. For $h \in A$ the value function can be split into two parts: the payoff that arises from following BAU until reaching the threshold $\tilde{h}$, and the subsequent payoff under stabilization. We state some intermediate results before discussing this two-part value function.

Define $T(h;\tilde{h})$ as the amount of time it takes to reach the stabilization threshold (the "time-to-go"), given the current state $h \in A$; T is the solution to

$$\tilde{h} = a - \left(a-h\right)e^{-\mu T} \Rightarrow \tag{A.7}$$

$$T\left(\tilde{h};\tilde{h}\right) = 0 \;\text{ and }\; \frac{dT}{dh} = \frac{-1}{\mu\left(a-h\right)}. \tag{A.8}$$

For $h \in A$ and for $\tau \leq T$

$$\frac{dy\left(\tau\right)}{dh} = \frac{d\int_0^\tau h(s)ds}{dh} = \int_0^\tau \frac{dh(s)}{dh}ds = \int_0^\tau e^{-\mu s}ds = \frac{1-e^{-\mu\tau}}{\mu}. \tag{A.9}$$

In addition, for $h \in A$ and for $\tau > T$

$$\frac{dy(\tau)}{dh} = \frac{d\left(\int_0^T h(s)ds + \tilde{h}(\tau - T)\right)}{dh} =$$
$$\int_0^T \frac{dh(s)}{dh} ds + \left(h(T) - \tilde{h}\right) \frac{dT}{dh} = \int_0^T e^{-\mu s} ds$$

The last equality uses the fact that $h(T) = \tilde{h}$, from the definition of T. Using equation (A.7) and (A.8), we can invert the function $T(h; \tilde{h})$ to write the initial condition h as a function of the time-to-go T and the threshold $\tilde{h}$. Using this fact, equation (A.9) and the definition of $y(\tau)$, we have

$$\begin{aligned} y(T) &= \int_0^T h(s)ds \Rightarrow \\ \frac{dy(T)}{dT} &= h(T) + \int_0^T \frac{dh(s)}{dh} \frac{dh}{dT} ds \end{aligned} \tag{A.10}$$

We now discuss the value function for $h \in A$. Splitting the payoff into the parts before and after the threshold is reached, this function equals

$$V\left(h; \tilde{h}\right) = \int_0^T e^{-y(\tau)} \theta(\tau) U(0) dt + \int_T^\infty e^{-y(\tau)} \theta(\tau) U(1) dt$$

and its derivative with respect to h (using equation (A.9)) is

$$V_h\left(h; \tilde{h}\right) = (U(0) - U(1)) e^{-y(T)} \theta(T) \frac{dT}{dh} +$$
$$\int_0^T \frac{d\left(e^{-y(\tau)}\right)}{dh} \theta(\tau) U(0) dt + \int_T^\infty \frac{d\left(e^{-y(\tau)}\right)}{dh} \theta(\tau) U(1) dt$$
$$= \frac{-(U(0) - U(1))}{\mu(a-h)} e^{-y(T)} \theta(T) -$$
$$\left(\int_0^T \left(\frac{1 - e^{-\mu\tau}}{\mu}\right) e^{-y(\tau)} \theta(\tau) U(0) dt + \int_T^\infty \left(\frac{1 - e^{-\mu T}}{\mu}\right) e^{-y(\tau)} \theta(\tau) U(0) dt\right).$$

Using this expression, we can write the optimality condition (A.6) as

$$U(0) - U(1) > (U(0) - U(1)) e^{-y(T)} \theta(T) +$$
$$\mu(a-h) \left(\int_0^T \left(\frac{1 - e^{-\mu\tau}}{\mu}\right) e^{-y(\tau)} \theta(\tau) U(0) dt + \int_T^\infty \left(\frac{1 - e^{-\mu T}}{\mu}\right) e^{-y(\tau)} \theta(\tau) U(0) dt\right). \tag{A.11}$$

It is convenient to treat T as the independent variable, recognizing that the initial condition h is a function of T (from equation (A.7)): $h = h(T)$. The existence of a set A with positive measure requires that inequality (A.11) holds for small positive values of T, i.e. for initial conditions h close to but smaller than $\tilde{h}$.

The first order Taylor expansion of the first term on the right side of inequality (A.11) is

$$(U(0) - U(1)) - (U(0) - U(1)) \left(\tilde{h} + r(0)\right) T + o(T). \tag{A.12}$$

This expansion uses equations (3) and (A.10) and the fact that $\theta(0) = 1$. Using the fact that $1 - e^{-\mu T} = 0$ at $T = 0$, the first order Taylor expansion of the second term on the right side of inequality (A.11) is

$$\begin{aligned} &\mu \left(a - \tilde{h}\right) T \textstyle\int_0^\infty e^{-y(\tau)} \theta(\tau) U(1) dt + o(T) = \\ &\mu \left(a - \tilde{h}\right) T \textstyle\int_0^\infty e^{-\tilde{h}\tau} \theta(\tau) U(1) dt + o(T) = \\ &\mu \left(a - \tilde{h}\right) T \tfrac{(1-\beta)\gamma + \beta\delta + \tilde{h}}{\left(\tilde{h}+\gamma\right)\left(\tilde{h}+\delta\right)} U(1) + o(T). \end{aligned} \tag{A.13}$$

Substituting expressions (A.12) and (A.13) into inequality (A.11), dividing by T and letting $T \to 0$ (from above) produces the inequality

$$(U(0) - U(1)) \left(\tilde{h} + r(0)\right) > \mu \left(a - \tilde{h}\right) \frac{(1-\beta)\gamma + \beta\delta + \tilde{h}}{\left(\tilde{h} + \gamma\right)\left(\tilde{h} + \delta\right)} U(1). \tag{A.14}$$

Using $x \equiv 1 - \frac{U(1)}{U(0)}$ and $r(0) = \beta\gamma + \delta(1-\beta)$ (from equation (3)), and replacing $\tilde{h}$ with h, inequality (A.14) can be expressed as

$$\frac{x}{1-x}(h + \beta\gamma + \delta(1-\beta)) > \mu(a-h) \left(\frac{\beta}{h+\gamma} + \frac{1-\beta}{h+\delta}\right)$$

or, equivalently,

$$x > \Theta(h),$$

where $\Theta(h)$ is defined in equation (26), establishing part (iii).

(iv) Using

$$-\xi'(h) = \int_0^\infty te^{-ht}\theta(t)dt = \frac{\beta}{(h+\gamma)^2} + \frac{1-\beta}{(h+\delta)^2},$$

we express $\pi(h)$, defined in (15), as

$$\pi(h) = \frac{1}{1+\mu(a-h)\left(\frac{\beta}{(h+\gamma)^2} + \frac{1-\beta}{(h+\delta)^2}\right)}.$$

Expanding $1-\pi(h)-\Theta(h)$ as a polynomial in β and collecting terms gives (after some algebraic manipulations) equation (28). ∎

Proposition 3: *Given the initial hazard $h \in [0,a]$, the optimal restricted-commitment policy is to stabilize if and only if $x \leq x^C(h)$. This policy is time consistent for all $h \in [0,a]$ and $x \in [0,1]$ if and only if $\frac{dx^c}{dh} \leq 0$. A sufficient condition for this inequality is $\mu \geq a+\delta+g(\eta-1)$.*

Proof. (i) This claim follows from differentiating the functions $\nu(h)$ and $\xi(h)$ and by inspection. (ii) We begin with

$$y^B(t,h) \equiv \int_0^t (a-(a-h)e^{-\mu\tau})d\tau = at-(a-h)\frac{1-e^{-\mu t}}{\mu}, \quad \text{(A.15)}$$

where $y^B(t,h)$ is a specialization of $y(0,t)$, defined in (8), when the hazard process under BAU evolves (following equation (4)) according to $h(t) = a-(a-h_0)e^{-\mu t}$. From equations (18), (17) and (A.15),

$$\nu(h)-\xi(h) = \int_0^\infty \theta(t)\left(e^{-y^B(t,h)} - e^{-ht}\right)dt. \quad \text{(A.16)}$$

It is easy to verify that $\frac{1-e^{-\mu t}}{\mu}$ is strictly decreasing in μ for $\mu > 0$ and equals t at $\mu = 0$. Therefore, $y^B(t,h) > ht$ when $h < a$ and $\mu > 0$, and the right-hand side of equation (A.16) is negative. (iii) This claim is merely a summary of the derivation in the text above equation (29).

(iv) (Sufficiency) Suppose that $\lambda(h)$ is non-decreasing. Then for any $1 - x \geq \lambda(h)$ it is optimal to stabilize. Since h does not change under stabilization, it is also optimal to stabilize at any point in the future. For any $1 - x < \lambda(h)$ it is optimal to follow BAU. Since h increases along the BAU trajectory, the inequality $1 - x < \lambda(h)$ continues to hold along this trajectory and BAU remains optimal. (Necessity). Suppose that λ is strictly decreasing over some interval $0 \leq h_1 < h < h_2 \leq a$. Choose a value of h in this interval (the initial condition $h(0)$), and choose $1 - x = \lambda(h(0)) - \epsilon$, where ϵ is small and positive. At this initial condition and for this value of $1 - x$, it is optimal to follow BAU, causing h to increase. Because λ is decreasing in this neighborhood, there is a future time $t > 0$ at which $1 - x = \lambda(h(t))$. At this time, it becomes optimal to stabilize, so the initial decision to pursue BAU in perpetuity is not time consistent.

(v) Using (21) and (22), we express $\lambda(h)$ as

$$\lambda(h) = \frac{\int_0^\infty e^{-y^B(t,h)}\theta(t)dt}{\int_0^\infty e^{-ht}\theta(t)dt}. \tag{A.17}$$

Using equation (A.15) we have

$$y_h^B(t,h) \equiv \partial y^B(t,h)/\partial h = \frac{1 - e^{-\mu t}}{\mu}. \tag{A.18}$$

The argument h in $y^B(t,h)$ is the initial hazard. Differentiating (A.17) with respect to h, we see that $\lambda'(h) > 0$ if and only if

$$\int_0^\infty e^{-y^B(t,h)}\theta(t)dt \int_0^\infty e^{-ht}t\theta(t)dt > \int_0^\infty e^{-ht}\theta(t)dt \int_0^\infty e^{-y^B(t,h)}y_h^B(t,h)\theta(t)dt. \tag{A.19}$$

Noting $\int_0^\infty e^{-ht}\theta(t)dt = \frac{\beta}{h+\gamma} + \frac{1-\beta}{h+\delta}$ and $\int_0^\infty e^{-ht}t\theta(t)dt = \frac{\beta}{(h+\gamma)^2} + \frac{1-\beta}{(h+\delta)^2}$ and using (A.18), we express (A.19) as

$$\begin{aligned}\left(\tfrac{\beta}{(h+\gamma)^2} + \tfrac{1-\beta}{(h+\delta)^2}\right) \int_0^\infty e^{-y^B(t,h)}\theta(t)dt > \\ \left(\tfrac{\beta}{h+\gamma} + \tfrac{1-\beta}{h+\delta}\right) \int_0^\infty e^{-y^B(t,h)}\theta(t)\tfrac{1-e^{-\mu t}}{\mu}dt.\end{aligned} \tag{A.20}$$

Since $\delta > \gamma$, the right-hand side of inequality (A.20) is smaller than

$$\left(\frac{\beta}{(h+\gamma)^2} + \frac{1-\beta}{(h+\delta)^2}\right) \int_0^\infty e^{-y^B(t,h)}\theta(t)\frac{(h+\delta)(1-e^{-\mu t})}{\mu}dt. \tag{A.21}$$

Thus, it suffices to show that the left-hand side of (A.20) exceeds (A.21), i.e., that

$$\int_0^\infty e^{-y^B(t,h)}\theta(t)\left(1 - \frac{(h+\delta)(1-e^{-\mu t})}{\mu}\right) dt > 0,$$

which is guaranteed to hold if $\mu > h + \delta$. Since $h \le a$ and h approaches a under BAU, the inequality holds at all $h \in [0, a]$ if $\mu > a + \delta$. ∎

Proposition 4: *Under constant discounting (with $\beta = 0$), it is optimal to stabilize in perpetuity when $x \le 1 - \sigma^0(h)$ and it is optimal to follow BAU in perpetuity when $x > 1 - \sigma^0(h)$. The function $\sigma^0(h)$ determines the boundary between the BAU and stabilization regions and $\pi^0(h)$ is irrelevant.*

Proof. We first point out that existence of a solution to the optimal control problem requires that $\sigma^0(h) \ge \pi^0(h)$ over $h \in [0, a]$. We then show that there is no solution to the regulator's optimization problem that involves delayed stabilization. We then show that stabilization is optimal if and only if $x \le 1 - \sigma^0(h)$.

If $\sigma^0(h) \ge \pi^0(h)$ over $h \in [0, a]$ were not satisfied, then (using the argument in the proof of Proposition 1) there would be some initial h and values $0 < \frac{U(1)}{U(0)} < 1$ for which there is no Markov perfect solution. However, the objective function under constant discounting is bounded and a solution to the optimal control problem exists. Therefore, $\sigma^0(h) \ge \pi^0(h)$.

Constant discounting occurs when $\beta = 0$ or $\beta = 1$ or $\gamma = \delta$. It is clear from equation (28) that condition (27) is not satisfied in any of these cases, implying, in view of Proposition 2 Part (iii), that there can be no equilibrium with delayed stabilization.

We now turn to the main part of the proof. For h close to but smaller than a, $\sigma^0(h) > \pi^0(h)$. (We established the weak inequality above; here we need the strict inequality.) This claim uses a Taylor expansion. The Taylor expansion uses the facts that $\sigma^0(a) = \pi^0(a) = 1$ and the derivatives evaluated at $h = a$:

$$\sigma_h^0(a) = \frac{\mu}{(a+\mu+\delta)(\delta+a)} < \frac{\mu}{(\delta+a)^2} = \pi_h^0(a).$$

Thus, for some parameter values and initial conditions, $\pi^0(h) < \frac{U(1)}{U(0)} < \sigma^0(h)$ holds. For parameters that satisfy this inequality, in view of Proposition 1, the DPE (30) admits two solutions. With constant discounting, however, the solution to the optimization problem is unique. The possibility that there are multiple solutions to the necessary condition (the DPE), even though there is a unique optimal policy, also occurs in other control problems [e.g., 29]. We use the same line of reasoning as in the "Skiba problem" to identify the optimal policy.

Consider the situation where $\pi^0(h) < \frac{U(1)}{U(0)} < \sigma^0(h)$. Denote $V^S(h)$ and $V^B(h)$ as the value functions that satisfy the DPE (30) under stabilization and BAU, respectively, and let $V(h) = \max\left\{V^S(h), V^B(h)\right\}$ denote payoff under the optimal decision. The arguments used in the proof of Proposition 1 imply that for $\frac{U(1)}{U(0)} < \sigma^0(h)$, $V^B(h)$ satisfies

$$\begin{aligned} V^B(h) &= \tfrac{1}{\delta+h}\max\left\{U(1), U(0) + \mu(a-h)V_h^B(h)\right\} \\ &= \tfrac{1}{\delta+h}\left(U(0) + \mu(a-h)V_h^B(h)\right) > \tfrac{1}{\delta+h}U(1). \end{aligned} \tag{A.22}$$

Similarly, for $\frac{U(1)}{U(0)} > \pi^0(h)$, $V^S(h)$ satisfies

$$\begin{aligned} V^S(h) &= \tfrac{1}{\delta+h}\max\left\{U(1), U(0) + \mu(a-h)V_h^S(h)\right\} \\ &= \tfrac{1}{\delta+h}U(1) \geq \tfrac{1}{\delta+h}\left(U(0) + \mu(a-h)V_h^S(h)\right). \end{aligned} \tag{A.23}$$

From (A.22) and (A.23) we see that $V^B(h) > V^S(h)$ when $\pi^0(h) < \frac{U(1)}{U(0)} < \sigma^0(h)$. Therefore, when $\pi^0(h) < \frac{U(1)}{U(0)} < \sigma^0(h)$ the (unique) optimal policy is BAU.

Again using the arguments in Proposition 1, $V^S(h)$ is the only solution to the DPE when $\frac{U(1)}{U(0)} > \sigma^0(h)$; when this inequality is satisfied, the optimal solution is to stabilize. $V^B(h)$ is the only solution when $\frac{U(1)}{U(0)} < \pi^0(h)$; when this inequality is satisfied, BAU is the optimal solution. By convention, we break the tie, which occurs when $\frac{U(1)}{U(0)} = \sigma^0(h)$, by choosing stabilization. ∎

B Additional numerical analysis

This section presents graphs (as functions of h) of $1-\pi$, x^c and Θ. For $x < 1-\pi$ there is a MPE with immediate stabilization, and for $x < x^c$ instant stabilization is the optimal policy under restricted commitment. For some parameter values $1-\pi > x^c$ for all h, in which case there are MPE with "excessive stabilization", relative to the restricted commitment baseline. For other parameter values $1-\pi < x^c$, in which case for some levels of the cost parameter x, there are no MPE that involve stabilization even though stabilization is optimal under restricted commitment. In other cases, the two curves cross.

The figures also show the graph of Θ, which is always less than $1-\pi$. In some cases, the distance between these graphs is large, so for a non-negligible range of parameter values there are MPE with delayed stabilization.

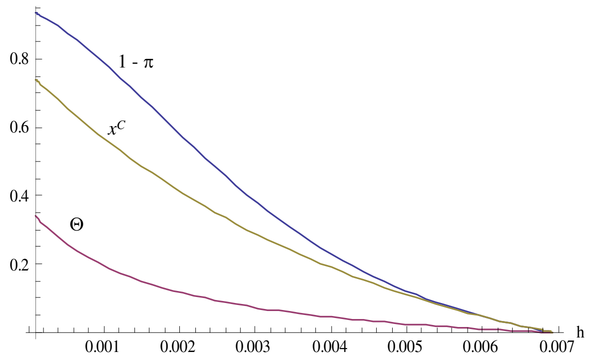

Figure B.1: Graphs of $1-\pi(h)$, $x^C(h)$ and $\Theta(h)$ when $g = 1\%$ and $\eta = 1.1$.

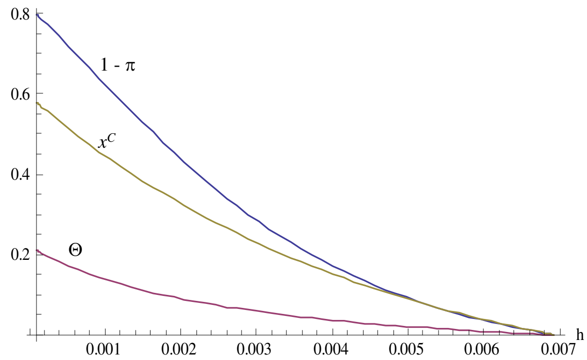

Figure B.2: Graphs of $1 - \pi(h)$, $x^C(h)$ and $\Theta(h)$ when $g = 2\%$ and $\eta = 1.1$.

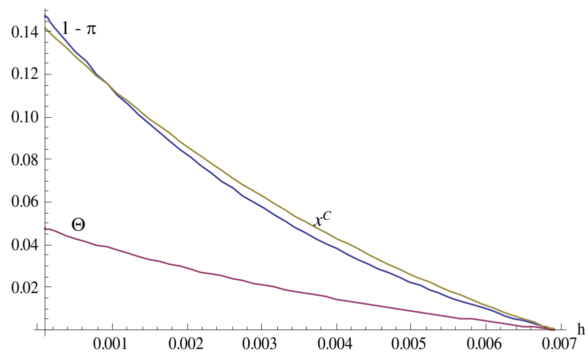

Figure B.3: Graphs of $1 - \pi(h)$, $x^C(h)$ and $\Theta(h)$ when $g = 1\%$ and $\eta = 2$.

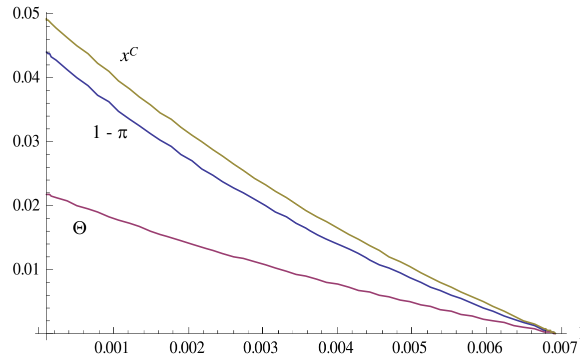

Figure B.4: Graphs of $1 - \pi(h)$, $x^C(h)$ and $\Theta(h)$ when $g = 2\%$ and $\eta = 2$.

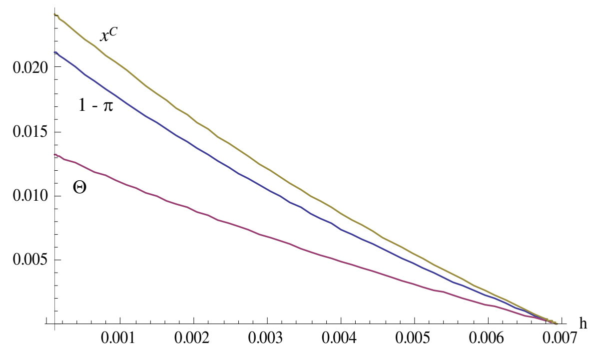

Figure B.5: Graphs of $1 - \pi(h)$, $x^C(h)$ and $\Theta(h)$ when $g = 1\%$ and $\eta = 4$.

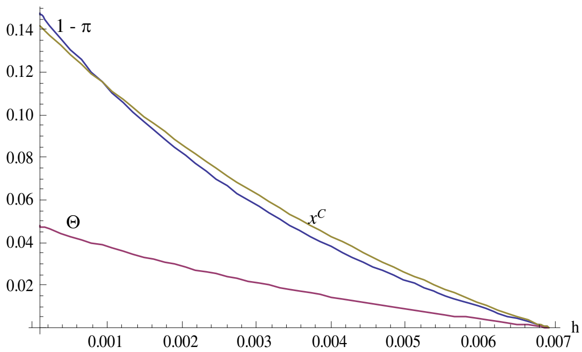

Figure B.6: Graphs of $1 - \pi(h)$, $x^C(h)$ and $\Theta(h)$ when $g = 2\%$ and $\eta = 4$.