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Notes, Comments, and Letters to the Editor

Non-constant discounting in continuous time☆

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Abstract

This paper derives the dynamic programming equation (DPE) to a differentiable Markov Perfect equilibrium in a problem with non-constant discounting and general functional forms. Beginning with a discrete stage model and taking the limit as the length of the stage goes to 0 leads to the DPE corresponding to the continuous time problem. The note discusses the multiplicity of equilibria under non-constant discounting, calculates the bounds of the set of candidate steady states, and Pareto ranks the equilibria.

JEL classification: D83; L50

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1. Introduction

This paper studies the set of differentiable Markov Perfect equilibria in a control problem with declining discount rates. The basic model is a discrete stage infinite horizon problem in which each stage lasts for ε units of time. The discount rate declines during the first $n = \frac{T}{\varepsilon}$ periods, where n is an arbitrarily large but finite constant. After n periods the discount rate becomes constant. I adapt the approach taken by Harris and Laibson [4] (where $n = 1$) for the general case where $n \geqslant 1$, to obtain a dynamic programming equation (DPE). A solution to this DPE, i.e. a value function and a control rule, is indexed by the parameter ε. Formally taking the limit as $\varepsilon \longrightarrow 0$ yields a continuous time DPE that corresponds to the continuous time limit of the discrete stage problem.

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A Markov equilibrium to the problem under non-constant discounting can be obtained by solving a control problem with constant discounting but a different objective function, jointly with a "side condition". The necessary condition to the DPE leads to a differential equation that the equilibrium control must satisfy, an Euler equation. This problem has an "incomplete transversality condition" (as in [11]) leading to non-uniqueness of candidate equilibria (those satisfying the necessary conditions). This source of non-uniqueness is different than in [8], which relies on decision rules that are step functions—and therefore not differentiable. The construction of an equilibrium here assumes differentiability. The Markov assumption eliminates the type of non-uniqueness that arises when strategies are supported by the use of threats, as in [9]. Under a plausible restriction, more conservative decision rules (i.e. those that result in a higher steady state of the resource) Pareto dominate less conservative rules.

2. The model and results

The instantaneous discount rate at time s is a non-increasing function $r(s)$ for $0 \leqslant s \leqslant T$, and $r(s) = \bar{r}$ (a constant) for $s \geqslant T$. The discount factor at time t used to evaluate a payoff at time $\tau + t$, $\tau \geqslant 0$, is $\theta(\tau) = \exp\left(-\int_0^\tau r(s)\, ds\right)$. The state variable S evolves according to

$$\dot{S} = f(S, x), \tag{1}$$

where x is the control variable. The flow of payoff is the concave function $U(S, x)$, which is increasing in x. This problem has an infinite horizon, regardless of the value of T. The present discounted value of the steam of future payoffs evaluated at time t is

$$\int_0^\infty \theta(\tau) U(S_{t+\tau}, x_{t+\tau})\, d\tau. \tag{2}$$

I study this problem by taking the limit of a discrete stage approximation in which each period lasts for ε units of time, and all variables are constant within an interval. This approximation replaces Eq. (1) and the objective in (2) with

$$S_{t+\varepsilon} = f(S_t, x_t)\, \varepsilon, \quad \sum_{i=0}^{\infty} \theta(i\varepsilon)\, U(S_{t+i\varepsilon}, x_{t+i\varepsilon})\, \varepsilon. \tag{3}$$

For fixed $T < \infty$ and $\varepsilon > 0$, the number of periods during which the discount rate is non-constant is $n = \frac{T}{\varepsilon} < \infty$ (ignoring the "integer problem"). The fact that $n < \infty$ makes it possible to use an extension of a method described in [4] to obtain a rigorous and transparent derivation of the DPE. This DPE contains ε as an argument. Passing to the continuous time limit of this DPE is "formal" , in the sense that the procedure assumes that the endogenous functions are analytic in ε, so that the Taylor approximation is valid.

2.1. The dynamic programming equation

Let $\chi(S)$ be the equilibrium control rule (a function to be determined) for the continuous time limit of the discrete stage problem. Define $H(S) \equiv U(S, \chi(S))$, the flow of payoff under the equilibrium rule. In equilibrium $x_\tau = \chi(S_\tau)$, so the solution to Eq. (1) can be written as $S_{t+\tau} = g(\tau; S_t)$. The function $K(S)$ is defined as

$$K(S_t) \equiv \int_0^T \theta(\tau)\, (r(\tau) - \bar{r})\, H(g(\tau; S_t))\, d\tau. \tag{4}$$

Define $W(S_t)$ as the value function, i.e. as the equilibrium value of the payoff in expression (2) when the control rule $\chi(S)$ is used and the initial condition is S_t. The first result is:

Proposition 1. *The limit as* $\varepsilon \to 0$ *of the dynamic programming equation to the discrete stage equilibrium problem in* (3) *is*

$$K(S_t) + \bar{r}W(S_t) = \max_x \left\{ U(S_t, x) + W'(S_t) f(S_t, x) \right\}. \tag{5}$$

The parameter T (the amount of time during which the discount rate is falling) appears in the definition of the function K, in Eq. (4). Although a finite value of T is needed to begin the derivation that leads to the DPE, we can study the problem with $T = \infty$ by taking limits with respect to T. The presence of the function $K(S_t)$ in the DPE is due to non-constant discounting. The solution to the continuous time problem consists of a function $\chi(S)$ that maximizes $U + W'f$, and satisfies $K + \bar{r}W = H + W'f$ and Eq. (4).

The following is obvious from inspection of Eq. (5):

Remark 1. The MPE to the equilibrium problem with a non-constant discount rate is produced by solving the necessary conditions to the "auxiliary control problem" given by

$$\begin{aligned} &\max \int_0^\infty e^{-\bar{r}\tau} \left(U(S_{t+\tau}, x_{t+\tau}) - K(S_{t+\tau}) \right) d\tau \\ &\qquad \text{subject to } \dot{S} = f(S, x). \end{aligned} \tag{6}$$

In this problem, the decision-maker treats the function $K(\cdot)$ as exogenous, since this function is determined by the behavior of "future decision-makers", and the current decision-maker takes that behavior as given. Eq. (4) is the "side condition" alluded to in the Introduction.

I cannot appeal to any of the standard sufficiency conditions, because of the endogeneity of $K(S)$; therefore, the analysis relies on necessary conditions. [1]

To help interpret Remark 1, consider the case where S is wealth and x is consumption, with $U_S > 0$. If the MPE flow of utility is positive and increasing in wealth ($H > 0$ and $H' > 0$) and in addition $\frac{\partial g(\tau; S_t)}{\partial S_t} > 0$, then $K > 0$, and $K' > 0$. The presence of the negative stock amenity, $(-K)$ makes it less attractive to accumulate wealth.

A planner with commitment ability solves a non-stationary control problem with flow payoff $U(S, x)$. Since $\lim_{t\to\infty} r(s) = \bar{r}$, the necessary conditions to this non-stationary control problem are asymptotically equivalent to those of a control problem with utility $U(S, x)$ and constant discount rate $\bar{r}$ [10]. Denote this as the "asymptotic control problem". The asymptotic control problem and the auxiliary control problem differ only because of the presence of $-K(S)$ in the flow payoff in the latter. For $K > 0$ and $K' > 0$, under plausible conditions the optimal wealth trajectory for the asymptotic control problem is greater than the trajectory under the auxiliary control problem. [2] Thus, for sufficiently large t the wealth and consumption profiles are higher for the planner who is able to make binding commitments.

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[1] However, when $r(0) - \bar{r}$ is small, $K(S)$ and all of its derivatives are also small. Therefore, $U - K$ is concave when $r(0) - \bar{r}$ is sufficiently small and U is strictly concave in S. Given the other assumptions on U and f in the text, the necessary conditions are then sufficient.

[2] Sufficient conditions are: $U_{Sx} \equiv 0$, $f(S, x) = F(S) - x$ and $U_{SS} - K'' < 0$.

A planner would like to drive the state to the first-best level (the steady state of the asymptotic control problem), but achieving this requires commitment ability. The planner also wants to consume at a high rate in the short run (because $r(0)$ is large). Behaving as if she uses the long-run (low) discount rate, but suffers a negative stock amenity, captures these conflicting incentives.

The function K simplifies if $\theta(t)$ is a convex combination of exponential discount factors:

Remark 2. Let $\theta(t) = \beta e^{-\gamma\tau} + (1-\beta)\, e^{-\delta\tau}$, with $0 < \gamma < \delta$ and $0 < \beta < 1$, and let $T = \infty$. For this discount function, $\bar{r} = \gamma < \beta\gamma + (1-\beta)\,\delta = r(0)$ and

$$K(S_t) \equiv (\delta-\gamma)\,(1-\beta) \int_0^\infty e^{-\delta\tau} H(g(\tau; S_t))\, d\tau \Longrightarrow \tag{7}$$

$$\delta K(S) = (\delta-\gamma)\,(1-\beta)\, H(S_t) + K'(S) f(S, \chi(S)). \tag{8}$$

The MPE is a solution to the coupled Eq. (5) (with $\bar{r} \equiv \gamma$) and (8). The two differential equations are based on discounting at the rates γ and δ, respectively.

The "transversality condition at infinity" for the problem in (6) is $\lim_{t\to\infty} \theta(t)\, W'(S_t) = 0$ [6]. I assume that this condition is satisfied by requiring that the state converge to a steady state, a solution to $f(S, \chi(S)) = 0$. Asymptotic stability of a steady state, S_∞, requires

$$z(S_\infty) \equiv f_S + f_x \chi' < 0 \quad \text{where } f\,(S_\infty, \chi\,(S_\infty)) = 0. \tag{9}$$

Standard manipulations of Eq. (5) lead to the Euler equation. A candidate steady state S_∞ is a value of S that satisfies both the Euler equation (not shown) evaluated at the steady state, and the stability condition, Eq. (9). For a particular steady state S_∞ that satisfies both of these conditions, define $\hat{\chi}\,(S; S_\infty)$ as the function that solves the Euler equation and that drives the state to S_∞ (thereby satisfying the transversality condition at infinity). The function $\hat{\chi}\,(S; S_\infty)$ is a "candidate" equilibrium, since—as noted above—it is constructed using only the necessary conditions.

The problem with non-constant discounting has an "incomplete" transversality condition, leading to multiplicity of equilibria, for essentially the same reason as in the differential game studied by Tsutsui and Mino [11]. The equilibrium problem with non-constant discounting is equivalent to a game amongst a succession of planners, each of whom chooses the control variable for a short (or infinitesimal) amount of time. The equilibrium conditions, together with the assumptions of Markov perfection and differentiability of the control rule, do not provide enough information to identify agents' beliefs regarding the response of other agents to a departure from the steady state. That is, the equilibrium conditions do not identify $\chi'\,(S_\infty) \equiv \hat{\chi}_1\,(S_\infty; S_\infty)$. Consequently, these conditions do not identify a unique steady state. However, the stability requirement puts bounds on $\chi'(S_\infty)$, as discussed in the next subsection. [3]

[3] The policy function $\hat{\chi}\,(S; S_\infty)$ may be defined only for a subset of state space. A similar circumstance arises in differential games where the incomplete transversality condition leads to a continuum of candidate decision rules. In a linear-quadratic setting, Tsutsui and Mino [11] show that only the linear equilibrium is defined for the entire real line. The requirement that the equilibrium be defined for all of state space is likely to substantially reduce the multiplicity discussed in the next subsection. However, for some one-dimensional economic problems it is reasonable to define the state space as a subset of the real line. In that case, the requirement that the equilibrium be defined for the entire state space need not reduce the set of candidates to a singleton.

2.2. A specialization

The special case $U_S \equiv 0$ and $f(S,x) = F(S) - x$ (i.e., $f_x \equiv -1$) can be used to compare the necessary condition under constant and non-constant discounting. This specialization is appropriate in a resource problem where the flow of utility is independent of the stock, and consumption reduces the stock linearly. In this special case, standard manipulations of the DPE, Eq. (5), result in the Euler equation (a "modified Ramsey Rule"):

$$\eta(x_t)\frac{\dot{x}}{x} = F'(S_t) - \left(\bar{r} + \frac{K'(S_t)}{U'(x_t)}\right), \tag{10}$$

where $\eta = -\frac{U''}{U'}x$ is the elasticity of marginal utility, and $\bar{r} + \frac{K'(S_t)}{U'(x_t)}$ is the effective discount rate. If the discount rate is constant ($r(s) \equiv \bar{r}$ for all s) then $K(S) \equiv K'(S) \equiv 0$ and Eq. (10) reproduces the standard Ramsey Rule. With non-constant discounting (in a MPE), the "current regulator" takes into account the effect of current consumption on future values of the state, and the resulting changes in future consumption and in future marginal utility. The function $K'(S)$ incorporates these reactions by "future regulators".

In variations of this model, Barro [1] and Cropper and Laibson [3] note that the (linear) Markov equilibrium is observationally equivalent to the equilibrium under constant discounting. That is, there is a constant discount rate that gives rise to the same equilibrium control rule as the linear Markov rule under non-constant discounting. It is well known that observational equivalence (between constant and non-constant discounting) does not hold in general. From Eq. (10), observational equivalence holds if and only if $\frac{K'(S_t)}{U'(x_t)}$ is a constant.

The following proposition characterizes the set of candidate equilibria. To simplify notation, write $r_0 = r(0)$ and $z = z(S_\infty)$, defined in Eq. (9) in addition, define $g \equiv F'(S_\infty)$ and $\lambda = \left(\int_0^\infty e^{-\lambda\tau}\,d\tau\right)^{-1} \equiv \left(\int_0^\infty \theta(\tau)\,d\tau\right)^{-1}$, where $\bar{r} < \lambda < r_0$.

Proposition 2. *Let $U_S \equiv 0$ and $f(S,x) = F(S) - x$ with $F(S)$ strictly concave. Assume that in equilibrium the state converges to an interior steady state.* (i) *Any value S_∞ that satisfies*

$$F'(S_\infty) = \bar{r} + \chi'(S_\infty)\int_0^T \theta(\tau)\,(r(\tau) - \bar{r})\,e^{z\tau}\,d\tau \tag{11}$$

and the stability condition, *Eq.* (9) *is a candidate* (*interior*) *steady state*. (ii) *There exists either no candidate interior steady states*, *or there exists a continuum of candidate steady states*; *in the latter case*, *there exists a continuum of candidate equilibrium control rules $\hat{\chi}(S; S_\infty)$*. (iii) *For $T = \infty$, the steady state condition* (11) *simplifies to*

$$Q(g,z) \equiv 1 - (g - z)\left(\int_0^\infty \theta(\tau)\,e^{z\tau}\,d\tau\right) = 0. \tag{12}$$

At every candidate state $\chi'(S_\infty) > 0$ and

$$\bar{r} < F'(S_\infty) < r_0. \tag{13}$$

(iv) *Let $q(z)$ be a root of $Q(q(z), z) = 0$, defined in Eq.* (12). *If $\frac{dq}{dz} < 0$ for $z < 0$, then any interior stable steady state must satisfy $\lambda < F'(S_\infty) < r_0$.*

With constant discounting, the second term on the right-hand side of (11) vanishes, leaving the standard steady state condition $F'(S_\infty) = \bar{r}$. This equation determines a unique interior S_∞ if

$F'(0) > \bar{r}$. For concave U, the unique solution to the control problem is the consumption rule that satisfies the Euler equation and drives the state to S_∞. The transversality condition provides the boundary condition to the Euler equation.

In contrast, with non-constant discounting, the steady state condition (11) does not identify a unique steady state. The right-hand side of this equation contains the undetermined function $\chi'(S_\infty)$. We do not have "outside" information that identifies the value of $\chi'(S_\infty)$. Consequently, there exist a continuum of candidate equilibria.

Eq. (13) is a necessary condition for the steady state. Proposition 2(iv) tightens the lower bound if the function $q(z)$ is monotonic: in this case, any differentiable interior Markov perfect stable steady state can be supported by a model with a constant discount rate that is strictly greater than λ.

The monotonicity of $q(z)$ can be checked for specific discount functions. For example, if θ is the convex combination of exponential discount factors given in Remark 2, then

$$\lambda = \frac{\delta\gamma}{(1-\beta)\,\gamma + \beta\delta} \quad \text{and} \quad q(z) \equiv \frac{((1-\beta)\,\delta + \beta\gamma)\,z - \delta\gamma}{z - (1-\beta)\,\gamma - \beta\delta} \Longrightarrow \frac{dq}{dz} < 0.$$

If the regulator at an arbitrary time were able to make commitments, a positive steady state to her problem satisfies $F'(S_\infty) = \gamma$. Since $\gamma < \lambda$, all candidate MPE steady states are strictly below the steady state chosen by the regulator who can make commitments. [4]

Proposition 2 requires that $F(S)$ is strictly concave. If F is linear, i.e. if $F'(S) = g$, a constant, (and maintaining the restriction $U_S \equiv 0$) then the steady state condition (Eq. (11)) involves a single unknown. [5] With exponential discounting, the controlled system does not converge to an interior steady state, except for the knife-edge case where $g = r$, the constant discount rate.

The MPE with hyperbolic discounting can be obtained by solving a control problem with constant discount rate and the stock amenity function $-K(S)$. The inclusion of the function K means that (even for linear F and $U_s \equiv 0$) there are candidate interior stable steady states for range of values of g. If the function $q(z)$ is monotonic, there is at most one value of χ' that solves Eq. (11). For linear $F(S)$, denote the root to Eq. (11) as χ'^*, assuming that it exists, and denote $z^* = g - \chi'^*$. For the function $\theta(t)$ given in Remark 2, a straightforward calculation establishes that $z^* < 0 \iff \lambda < g < r_0$. For any g that satisfies this inequality, there is a unique χ'^*. In this case, the steady state condition identifies the value of χ'^*, but not the steady state S_∞.

2.3. Pareto ranking the steady states

The equilibrium problem with non-constant discounting is equivalent to a game amongst a succession of agents, each of whom wants to maximize the present discounted value of current and future welfare. (Agents do not care about the past.) An agent is indexed by the time at which she makes the decision. The agent at time t discounts utility at time $t+\tau$, $\tau \geqslant 0$, using the discount factor $\theta(\tau)$. An equilibrium is Pareto Efficient, at time t, if no other decision rule gives all current

[4] If $r_0 > F'(0) > \lambda$ the resource might be exhausted or preserved in the steady state. If $F'(0) < \lambda$ and $F(0) = 0$, the unique steady state is $S_\infty = 0$. In this case, the existence of a unique steady state implies a unique solution to the Euler equation, i.e. a unique MPE. In this situation, the problem has a complete transversality condition (i.e. it has a "natural boundary condition").

[5] If S is an argument of the utility function and $F(S)$ is linear, the steady state condition still involves two unknowns. Karp [7] discusses a discrete time quasi-hyperbolic example of this case. That paper also demonstrates a special case of Proposition 3, below.

and future agents a higher payoff. In the setting here, an equilibrium is Constrained Pareto Efficient if no other differentiable Markov Perfect decision rule gives all current and future agents a higher payoff.

Given a (candidate) equilibrium control rule $\hat{\chi}(S; S_\infty)$, the equilibrium payoff of an agent when the current state is S is the value function $W(S; \hat{\chi}(S; S_\infty))$, the solution to Eq. (5). Here I show explicitly the dependence of this value function on the equilibrium decision rule, indexed by the steady state toward which this decision rule drives the state. The following definitions simplify the discussion:

$$\omega(S; S_\infty) \equiv W(S; \hat{\chi}(S; S_\infty)), \quad \omega_2(S; S_\infty) = \frac{\partial \omega(S; S_\infty)}{\partial S_\infty}.$$

Define Ψ as the open set of candidate steady states. If the monotonicity condition in Proposition 2(iv) holds, the supremum of Ψ is $\max\left\{0, F'^{-1}(\lambda)\right\}$. For the example where $\theta(t) = \beta e^{-\gamma\tau} + (1-\beta)\, e^{-\delta\tau}$, we have $\Psi = \left(F'^{-1}(r_0),\ F'^{-1}(\lambda)\right)$, provided that $F'^{-1}(r_0) > 0$.

An equilibrium rule $\hat{\chi}\left(S; S_\infty^*\right)$ is Constrained Pareto Superior to another rule $\hat{\chi}\left(S; S_\infty^\#\right)$ if and only if both S_∞^* and $S_\infty^\#$ are elements of Ψ and

$$\omega(S; S_\infty^*) \geqslant \omega(S; S_\infty^\#) \tag{14}$$

for all initial conditions S and strict inequality holds for some S.

Inequality (14) is difficult to check because it must hold for all S. A local condition that is simple to check provides a necessary condition for Constrained Pareto Efficiency. The local condition checks whether the current agent and all her successors would be willing to switch from a reference decision rule $\hat{\chi}(S; S_\infty)$ to a neighboring rule, given that the current state is S_∞. If all agents (current and future decision-makers) would be willing to make the switch, the reference decision rule is not Constrained Pareto Efficient

The state trajectory is monotonic in the neighborhood of the steady state ($z < 0$); if we switch from one decision rule to a neighboring rule, the state adjusts monotonically to the new steady state. Therefore, the local condition for Pareto ranking requires checking whether inequality (14) holds for values of S between the steady states of the reference and the neighboring decision rules, rather than for all possible values of S.

Clearly, a rule that is *less conservative* (i.e., that involves higher consumption for a given value of the state variable) than the reference rule always harms agents sufficiently far in the future; those future agents consume at a lower steady state level (relative to consumption under the reference rule). Therefore, a rule that is less conservative than an arbitrary candidate cannot Pareto dominate that candidate. [6]

A perturbation to a *more conservative* rule benefits agents sufficiently far in the future, since they consume at a higher level, as a result of the higher steady state. The current agent and her nearby successors have lower consumption but they appreciate the higher welfare of their successors, so it is not clear whether they benefit from the switch to the more conservative rule. The decision rule

[6] Caplin and Leahy [2] point out that the solution to a time-consistent dynamic optimization problem is one of a continuum of Pareto Efficient outcomes. With hyperbolic discounting, the MPE is not Pareto Efficient. Nevertheless, we can Pareto-rank the MPE.

$\hat{\chi}(S; S_\infty)$ is "locally Constrained Pareto dominated" by a more conservative neighboring rule if and only if $\omega_2(S_\infty; S_\infty) > 0$. Using this definition, we have the following:

Proposition 3. *For $U_S \equiv 0$, $f(S,x) \equiv F(S) - x$, $F'' < 0$ and $T = \infty$*

$$sign(\omega_2(S_\infty; S_\infty)) = sign\left(F'(S_\infty) - \lambda\right). \tag{15}$$

Any equilibrium with a steady state that satisfies $F'(S_\infty) > \lambda$ is locally Constrained Pareto dominated by an equilibrium with a higher steady state. If the monotonicity condition in Proposition 2(iv) *holds, the Pareto ranking of the MPE is identical to the ranking of the steady states of these equilibria: more conservative rules are always locally Pareto superior to less conservative rules.*

Proposition 2(iv) and Proposition 3 provide conditions under which Pareto dominance selects a conservative rule. We cannot speak of the "most conservative" MPE since Ψ is an open set. Pareto dominant rules imply slower asymptotic convergence to the steady state. That is, a lower value of $|z|$ implies a lower value of the effective steady state discount rate (under the monotonicity property of Proposition 2(iv)) and this implies a higher steady state.

2.4. Relation to previous literature

Barro [1] and Harris and Laibson [5] study continuous time models of non-constant discounting for the special case where $U_S \equiv 0$ (as in Section 2.2). Both of those papers assume that the equation of motion is linear in the state (contrary to the assumption in Proposition 2).

In Barro [1], the representative agent solves a *non-stationary* control problem, with $\dot{S} = w(t) + g(t)S - x$; $w(t)$ is the equilibrium wage rate and $g(t)$ is the equilibrium rental rate. In Barro's model, the functions $w(t)$, $g(t)$ depend on the economy-wide level of the stock, $S^e(t)$. The representative agent takes $S^e(t)$ as exogenous and therefore takes $w(t)$, $g(t)$ as exogenous. In equilibrium, $S(t) \equiv S^e(t)$. Thus, although the agent treats $w(t)$, $g(t)$ as exogenous, their equilibrium values are endogenous to the model. The agent in my setting solves a stationary problem, so the results in the two papers are not directly comparable. However, in the steady state to Barro's model, w and g are constants. Mutual consistency of the two models requires that Barro's steady state level of the stock must be supported as a steady state in my model, when I use his steady state values of w and g.

Barro sets $U(x) = \ln x$ and he selects a linear equilibrium; $T = \infty$ in his model. He shows that the linear Markov equilibrium is observationally equivalent to the equilibrium in a model with the constant discount rate λ. (See the definition of λ above and Barro's equation (13); his discount factor is $e^{-\rho\tau + \phi(\tau)} = \theta(\tau)$ in my notation.) Thus, the steady state for Barro's linear MPE is $S_\infty^{\text{Barro}} = F'^{-1}(\lambda)$.

In the *stationary* control problem with $U(x) = \ln x$, $\dot{S} = w + gS - x$ and $T = \infty$, it is straightforward to show (using Eq. (5)) that there exists a linear equilibrium control rule if and only if $g = \lambda$. That is, my stationary model (with linear equation of motion) supports Barro's steady state if and only if $g = \lambda = F'\left(S_\infty^{\text{Barro}}\right)$. The two models are mutually consistent.

When the growth equation is strictly concave (and in equilibrium the system converges to a steady state) Proposition 2(iv) shows (for monotonic $q(z)$) that S_∞^{Barro} is the supremum of candidate steady states. Proposition 3 implies that a decision rule that drives the state to S_∞^{Barro} locally Pareto dominates all MPE candidates. That is, at least in the neighborhood of the steady state, agents do better when they ignore the dependence (on the capital stock) of the wage and rental rate.

This result has a familiar economic explanation. By Eq. (12), $\chi'(S_\infty) > 0$, so actions are strategic substitutes near the steady state. Future consumption is too high, relative to the full-commitment level (the steady state of which satisfies $F(S) = \bar{r}$). Therefore, somewhat perversely, an agent has an incentive to increase current consumption in order to reduce the state and thereby restrain future consumption. An agent who treats the future wage and rental rate as exogenous has less strategic incentive to try to manipulate future behavior (by changing future values of the state variable). Since the strategic incentive is perverse, the diminished strategic incentive associated with price-taking behavior leads to a better outcome.

Harris and Laibson [5] use a *stationary* model with a linear (stochastic) equation of motion. In their setting, the "current" decision-maker exercises control for τ units of time, where τ is a random variable with exponential density. For the limiting case (an infinite hazard rate), their Theorem 17 shows that there is a time-consistent control problem (i.e., one with exponential discounting) that has the same value function as the original (non-exponential discounting) model. This time-consistent control problem is constructed using an "auxiliary" utility function that I denote as $\hat{U}(S, x)$. This function (in general) has the state as an argument, even though the primitive utility function does not. ($U_S \equiv 0$ in my notation.) Even for $U_S \equiv 0$, the function $K(S)$ defined in Eq. (4) is (in general) not a constant, so the utility function $U(x) - K(S)$ that appears in the control problem in (6) has the state variable as an argument. Thus, my Remark 1 has some similarity to Theorem 17 in [5].

However, there is an important difference between the two results. Harris and Laibson are able to construct the auxiliary utility function from the primitives of the problem. My analog to their auxiliary utility function ($U - K$), in contrast, involves the unknown function *K*; this function must be calculated as part of the equilibrium. Harris and Laibson's ability to construct $\hat{U}$ from primitives is critical to their uniqueness proof; their construction of $\hat{U}$ uses the limiting case where the "current" decision-maker's tenure is arbitrarily short (the hazard rate approaches infinity.) It appears to be an open question whether uniqueness holds in their model if the hazard rate is finite.

Appendix. Proof of propositions

Proof of Proposition 1. I begin by finding the DPE for the discrete time model. I then take the formal limit to obtain the DPE that corresponds to the continuous time model. The discrete stage problem uses the payoff and the equation of motion defined in Eq. (3). Define $\theta_i = \theta(\varepsilon i)$ and define $H_i = H(S_{t+i\varepsilon}) = U(S_{t+i\varepsilon}, \chi(S_{t+i\varepsilon}))$, where $\chi(\cdot)$ is a function to be determined. Each period lasts for ε units of time, so if the current calendar time is *t*, the calendar time *j* periods in the future is $t + j\varepsilon$. The discount rate decreases for the first *n* periods and is thereafter constant. For $i = 1, 2, \ldots, n-1$, define

$$\theta_{n-i} V_i \left(S_{t+(n-i)\varepsilon}\right) = \theta_{n-i} H_{n-i} \varepsilon + \theta_{n-i+1} V_{i-1} \left(S_{t+(n-i+1)\varepsilon}\right) \tag{16}$$

and

$$\theta_n V_0 \left(S_{t+n\varepsilon}\right) = \theta_n H_n \varepsilon + \theta_{n+1} V_0 \left(S_{t+(n+1)\varepsilon}\right). \tag{17}$$

The function $\theta_{n-i} V_i(\cdot)$ is the present value discounted back to time *t* (the "current time") of the equilibrium continuation payoff from time $t + (n-i)\varepsilon$ onwards. For periods $n, n+1 \ldots$ the discount rate is constant, so the value function in Eq. (17) is stationary; the index 0 on the value function denotes that there are 0 periods to go before the value function becomes stationary. At each of the stages $1, 2 \ldots n-1$ periods in the future, the subsequent discount rate changes, so the value functions in these periods also change. The subscript on the functions V_i in Eq. (16) denote the number of subsequent periods during which the discount rate will be non-constant. Use these two equation to "solve backwards" to obtain the relation

$$\theta_1 V_{n-1}\left(S_{t+\varepsilon}\right)=\sum_{i=1}^{n}\theta_i H_i\varepsilon+\theta_{n+1}V_0\left(S_{t+(n+1)\varepsilon}\right). \tag{18}$$

The DPE at period t is

$$W\left(S_t\right)=\max_x\left\{U\left(S_t,x\right)\varepsilon+\theta_1 V_{n-1}\left(S_{t+\varepsilon}\right)\right\} \tag{19}$$

and the maximized DPE is

$$W\left(S_t\right)=H_i\varepsilon+\theta_1 V_{n-1}\left(S_{t+\varepsilon}\right). \tag{20}$$

Some manipulations lead to the DPE for the discrete stage problem:

$$W\left(S_t\right)=\max_x\left\{U\left(S_t,x\right)\varepsilon+\sum_{i=1}^{n}\left(\theta_i-e^{-\bar{r}\varepsilon}\theta_{i-1}\right)H_i\varepsilon+e^{-\bar{r}\varepsilon}W(S_{t+\varepsilon})\right\}. \tag{21}$$

Taking the limit of the discrete time DPE as $\varepsilon\to 0$ leads to Eq. (5). □

Proof of Proposition 2. (i) Evaluating Eq. (10) at a steady state (where $\dot{x}=0$) gives the steady state condition

$$F'\left(S_\infty\right)=\bar{r}+\chi'\left(S_\infty\right)\int_0^T\theta\left(\tau\right)\left(r(\tau)-\bar{r}\right)\frac{\partial g(\tau;S_t)}{\partial S_t}\,d\tau. \tag{22}$$

In the neighborhood of the steady state we have $\dot{S}\approx z\left(S_t-S_\infty\right)\Rightarrow$

$$g\left(\tau;S_t\right)\approx e^{z\tau}S_t+S_\infty\left(1-e^{zt}\right)\Longrightarrow\frac{\partial g(\tau;S_t)}{\partial S_t}\approx e^{z\tau}, \tag{23}$$

where $z=F'\left(S_\infty\right)-\chi'\left(S_\infty\right)<0$ because of stability. Substituting Eq. (23) into (22) results in the steady state condition (11).

(ii) Eq. (11) is a single equation involving two unknowns and therefore generally has no solutions or a continuum of solutions. In the latter case, there are a continuum of candidate steady states. Associated with each steady state S_∞ is a function that satisfies the Euler equation, $\hat{\chi}\left(S;S_\infty\right)$, i.e. a candidate equilibrium.

(iii) For $T=\infty$, we have

$$\int_0^\infty\theta\left(\tau\right)r\left(\tau\right)e^{z\tau}\,d\tau=-\int_0^\infty\theta'\left(\tau\right)e^{z\tau}\,d\tau=1+z\int_0^\infty\theta\left(\tau\right)e^{z\tau}\,d\tau. \tag{24}$$

Using the definition of z and Eq. (24) we can rewrite Eq. (11) as

$$z-\bar{r}=\chi'\left[(z-\bar{r})\int_0^\infty \theta(\tau)\,e^{z\tau}\,d\tau\right]. \tag{25}$$

Eq. (25) implies Eq. (12). The fact that

$$\frac{1}{r_0-z}<\int_0^\infty \theta(\tau)\,e^{z\tau}\,d\tau<\frac{1}{\bar{r}-z},$$

implies Eq. (13).

(iv) Follows from differentiation of Eq. (12). □

Proof of Proposition 3. Define $z=F'(S_\infty)-\hat{\chi}_1(S;S_\infty)$. (Since here I show the steady state as an argument of χ, I write χ_1 instead of χ'.) Consider a perturbation of the control rule $\hat{\chi}(S;S_\infty)$ to $\hat{\chi}(S;S_\infty)-\varepsilon$ for small $\varepsilon>0$. This perturbation increases the steady state from S_∞ to $S_\infty-\frac{\varepsilon}{z}$. The function $g(\tau;S)$ is the solution to the equation of motion for S under a particular control rule. With some abuse of notation, here I use $g(\tau;S,\varepsilon)$ to denote the solution to the state equation under the perturbation. Beginning at the steady state S_∞ at time 0, the value of S at time $\tau\geqslant 0$ under this perturbation is [7]

$$S(\tau)=g(\tau;S_\infty,\varepsilon)\approx S_\infty-\frac{\varepsilon}{z}+\frac{\varepsilon}{z}e^{z\tau}. \tag{26}$$

For arbitrary initial condition S, the value of the program under the control rule $\chi(S;S_\infty)$ is $\omega(S;S_\infty)=\int_0^\infty \theta(\tau)\,U(\chi(\tau))\,d\tau$, with $\chi(\tau)\equiv\hat{\chi}(g(\tau;S);S_\infty)$. At $S=S_\infty$ the value of the program under the perturbation (i.e., replacing $\hat{\chi}(S;S_\infty)$ by $\hat{\chi}(S;S_\infty)-\varepsilon$) is

$$\begin{aligned}&\omega\left(S_\infty;S_\infty-\frac{\varepsilon}{z}\right)=\int_0^\infty \theta(\tau)\,U\left(\chi(\tau,\varepsilon)-\varepsilon\right)\,d\tau\\&\chi(\tau,\varepsilon)\equiv\hat{\chi}\left(g(\tau;S_\infty,\varepsilon);S_\infty\right).\end{aligned} \tag{27}$$

Taking derivatives (with respect to ε) of Eq. (27), and evaluating the result at $\varepsilon=0$, gives

$$\begin{aligned}\omega_2(S_\infty;S_\infty)&=-z\int_0^\infty \theta(\tau)\,U'\left(\chi_\infty\right)\left(\frac{\partial\chi}{\partial S}\frac{\partial g(\tau;S_\infty,\varepsilon)}{\partial\varepsilon}-1\right)d\tau\\&=-zU'\left(\chi_\infty\right)\int_0^\infty \theta(\tau)\left(\frac{\partial\chi}{\partial S}\frac{1}{z}\left(e^{z\tau}-1\right)-1\right)d\tau\\&=-zU'\left(\chi_\infty\right)\left(\frac{1}{z}-\left(\frac{\partial\chi}{\partial S}\frac{1}{z}+1\right)\int_0^\infty \theta(\tau)\,d\tau\right)\\&=-U'\left(\chi_\infty\right)\left(1-F'(S_\infty)\int_0^\infty \theta(\tau)\,d\tau\right).\end{aligned} \tag{28}$$

The second equality uses the approximation in Eq. (26); the third equality uses Eq. (12), and the fourth equality used the definition of z. Since $U'(x)>0$ by assumption, the last line of Eq. (28) implies Eq. (15). □

[7] It is more precise to replace the function z in Eq. (26) by $F'\left(S_\infty-\frac{\varepsilon}{z}\right)-\chi_1(S;S_\infty-\frac{\varepsilon}{z})$ since I am writing the approximation of the trajectory under the perturbed decision rule, not the approximation of the trajectory under the original decision rule. This correction complicates the notation but does not change anything of substance. When I evaluate derivatives at $\varepsilon=0$ the formulation in Eq. (26) and the correction in this footnote lead to the same result.

References

[1] R. Barro, Ramsey meets Laibson in the neoclassical growth model, Quart. J. Econ. 114 (1999) 1125–1152.
[2] A. Caplin, J. Leahy, The social discount rate, J. Polit. Econ. 112 (61) (2004) 1257–1268.
[3] M. Cropper, D. Laibson, The implications of hyperbolic discounting for project evaluation, in: P.R. Portney, J. Weyant (Eds.), Discounting and Intergenerational Equity, Resources for the Future, 1999.
[4] C. Harris, D. Laibson, Dynamic choices of hyperbolic consumers, Econometrica 69 (5) (2001) 935–957.
[5] C. Harris, D. Laibson, Instant gratification, Cambridge University Working paper, Econometrica, 2004, forthcoming.
[6] T. Kamihigashi, Necessity of transversality conditions for infinite horizon problems, Econometrica 69 (2001) 995–1012.
[7] L. Karp, Global warming and hyperbolic discounting, J. Public Econ. 8 (2005) 261–282.
[8] P. Krusell, A. Smith, Consumption-saving decisions with quasi-geometric discounting, Econometrica 71 (1) (2003) 365–375.
[9] D. Laibson, Self control and saving, Harvard Universty Working paper, 1994.
[10] C.Z. Li, K.G. Lofgren, Renewable resources and economic sustainability: a dynamic analysis with heterogeneous time preferences, J. Environ. Econ. Manage. 40 (2000) 236–250.
[11] S. Tsutsui, K. Mino, Nonlinear strategies in dynamic duopolisitc competition with sticky prices, J. Econ. Theory 52 (1990) 136–161.