Consumer Demand with Price Aggregators and Low-Rank Cross-Price Effects

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Abstract

Estimating consumer demands is a bread-and-butter undertaking in applied economics. In general, demand for each good depends on the prices of all goods and services, but for most applications it is impractical to estimate models of such high dimension. In this paper, we consider consumer demand with a low rank of the matrix of cross-price effects. We show that imposing a low rank is equivalent to introducing functions that we call "aggregators", where each aggregator maps information from an arbitrarily large vector of prices (and perhaps income) into a scalar. We then provide a complete characterization of the preferences that rationalize demand systems with such aggregators. These results have applications in a broad range of fields in economics. Most commonly-used demand systems (including directly-additive, indirectly-additive, non-homothetic CES and Kimball preferences) can be described with one or two of such aggregators where the price index may coincide with one of the aggregators. Nested and mixed logit require as many aggregators as nests or consumer types. Aggregators can also be naturally expressed as a function of observed product attributes. Using barcode data on yogurt purchases, we illustrate how to estimate a simple yet flexible specification of such a demand system with K aggregators, with or without using information on product attributes.

Keywords: Preferences; Cross-price effects; Generalized Separability; Latent variables; Low-rank approximation.

1 Introduction

Estimating consumer demands is a bread-and-butter undertaking in applied economics. Any such undertaking must somehow contend with the fact that, in general, demand for any good will depend on the prices of all goods and services. However, it is highly impractical to estimate models of such high dimension of price interactions, and so researchers invariably reduce dimension, often implicitly via a choice of functional forms. Examples include nested and mixed logit which extend the basic logit specification in order to generate more flexible cross-price effects by grouping goods or combining heterogeneous types of consumers.[1] These functional form assumptions lead to strong restrictions in terms of income effects (typically assuming quasi-linearity or homotheticity) and cross-price effects (e.g. by relying on predetermined groups of goods or assumptions on the patterns of consumer heterogeneity). Of course sometimes these limitations are features: we may need to restrict price and income effects to be able to construct a representative consumer, or we lack the data to estimate important patterns of substitution, but we may have lost an appreciation for the critical limitations of these functional forms and we lack a unifying perspective.

Our goal is to provide a disciplined approach to introduce more or less flexibility in functional forms of demand, depending on the data available and the needs of the researcher. In this paper we provide such an approach through the introduction of functions we call "aggregators". An aggregator maps information from an arbitrarily large set of prices (and perhaps income) into a scalar, thus summarizing information on many prices. A price index is a familiar example of an aggregator, but aggregators need not satisfy the usual requirements of a price index (for example, aggregators need not be homogeneous in prices). In a first step, we show that the number of aggregators coincides with the rank of the matrix of cross-price effects, adjusting for own-price effects, and provides a key metric that captures the complexity of a demand system. For instance, when the rank of cross-price effects is one, all cross-price effects can be captured by a single aggregator and demand for each good can then be expressed as a function of its own price and this aggregator.

This notion of rank of cross-price effects is also tightly linked to the complexity of the estimation problem and the number of parameters to identify, akin to low-rank approximations in machine learning. With a general demand system without rank restrictions, the complexity of cross-price effects grows as a quadratic function of the number of goods, and the estimation of cross-price effects becomes impossible with datasets covering a large number of goods or product varieties. By imposing a parameterization with low-rank cross-price effects, estimation with large datasets remains manageable as the number of parameters to be estimated grows at most linearly with the number of goods.

In a second step, we further assume that demand is rational, i.e., derived from the maximization of a utility function. We establish that rational demands with K aggregators must then take particular functional forms, and we show what these forms must be. In parallel, we show how integrating these demands gives rise to a rationalizing utility function, and thus provide a complete characterization of

[1] See, among others, Boyd and Mellman (1980), Berry (1994), Berry, Levinsohn, and Pakes (1995), McFadden and Train (2000).

the preferences which rationalize a demand system having K aggregators. What we call K-aggregator preferences and demands nest many of the preferences and demand structures existent in the literature, but also include novel forms. As a practical example, we provide a specification that allows for flexible non-parametric own-price effects combined with parametric cross-price effects captured by a symmetric matrix of any given rank K.

We begin with the homothetic case, focusing on cross-price effects, but then we also consider two ways to model non-homothetic preferences. In one approach, we consider Hicksian demand as a function of both prices and utility, which remains homogeneous of degree one in prices. In a second approach, we consider demand as a function of normalized prices, i.e., prices divided by total expenses (income), and under some additional assumptions we provide an explicit expression for indirect utility as a function of prices and income. In all cases, it is not hard to account for non-homotheticities while keeping similar functional forms.

Finally, we demonstrate how useful our approach can be in practical applications. We pursue three separate ideas. First, we show how demands in many recent applications can be viewed as demand systems of a particular rank. Second, if one has solid *a priori* beliefs about the rank of the demand system and the form of the aggregators, then we show by example how this information can be exploited in estimation. Third, if one does not have strong prior beliefs regarding K, we provide an algorithm which permits one to infer K while at the same time estimating the demand system.

Literature This paper primarily aims to contribute to the literature on modeling cross-price effects, which has a long tradition, not only in industrial organization (e.g. to understand the effects of competition) but also in other fields such as macroeconomics (e.g. cost-of-living estimation), international trade (e.g. in gravity equations), and development (understanding household consumption choices).

Earlier models of demand allowing for flexible cross-price effects include the Rotterdam model, PIGL, PIGLOG (and AIDS, developed by Deaton and Muellbauer 1980) and Translog. In most of these specifications, prices enter linearly or log-linearly, but are only valid demand systems at best in a local sense, not over the full range of prices and income. In this vein, EASI (Lewbel and Pendakur 2009) may be the latest and most flexible. For any specific good, it allows for flexible Engel curves, and prices enter log-linearly as in AIDS. Estimation of these models either does not impose restrictions on the rank of cross-price effects but is limited to a few goods (see e.g. Lewbel and Pendakur 2009), or impose trivial cross-price effects (e.g. Fajgelbaum and Khandelwal 2016).

A simple but powerful way to generate non-trivial cross-price effects is to construct nests, allowing for different elasticities of substitution between vs. within different nests. Nested logit has been widely used in industrial economics (see e.g. Berry 1994) while nested CES is a standard specification in macroeconomics and international trade (see e.g. Berry 1994). These types of preferences generate low-rank cross-price effects where the rank corresponds to the number of nests. However, for estimation, these nests have to be specified ex ante, and impose a lot of structure on substitution patterns. A recent class of demand proposed by Fosgerau, Monardo, and De Palma (2024), "inverse product-differentiation logit", proposes a more flexible way to construct nests without relying on hierarchies.

Our approach here provides a generalization of nests while keeping the low rank of price substitution patterns.

The now standard specification in industrial organization is that of "BLP" (Berry, Levinsohn, and Pakes 1995), itself a form of mixed logit (e.g. McFadden and Train 2000). While each consumer is typically assumed to have logit demand, thus with simple rank-one cross price effects, the BLP approach obtains more complex substitution patterns for aggregate demand, where the rank of cross-price effects can be as large as the number of types of consumers that are lumped together. In practice, the BLP approach involves adopting statistical assumptions regarding the distribution of consumer values. It allows for complicated cross-price elasticities, but these are typically generated by ad hoc assumptions on heterogeneity of price coefficients and attributes in logistic model. It is also difficult to link the patterns of cross-price effects to the distributional parameters to be identified in these models. In particular, BLP estimators involve a non-linear inversion of expenditure shares to express those as a linear function of prices and average valuation of product attributes. This non-linear inversion must account for parameters governing the heterogeneity in tastes, and thus cross-price effect terms. In comparison, our specification also involves an inversion but only with respect to own-price effects. In our baseline specification, cross-price effects enter linearly.

Non-trivial income effects can also be incorporated in our demand systems with low-rank cross-price effects. A first approach is to consider Hicksian demand and the expenditure function, which remains homogeneous in prices, conditional on utility. Thus, using utility as an additional aggregator, we are able to incorporate very flexible income effects, and their interaction with prices. This approach can be used for instance to generalize demand as in EASI (Lewbel and Pendakur 2009), or AIDS (Deaton and Muellbauer 1980), and obtain specifications that are globally regular. Another approach involves expressing aggregators as a function of normalized prices (prices relative to income) instead of prices. We obtain results that are similar (up to multiplicative terms) to the homothetic case.

Note that our concept of rank sharply differs from the notion of income rank introduced by Lewbel (1991), with higher rank demand systems admitting more complicated Engel curves. Our notion of rank focuses on cross-price effects, and demand system with K aggregators (hence rank K) can have any rank in terms of income effects.[2]

In applied theory, it is also useful to reduce the dimensionality of cross-price effects in order to improve tractability, while maintaining the assumption of consumer rationality. With price aggregators, the characterization of equilibrium is reduced to examining a few variables instead of potentially many prices. Focusing on a few aggregators also speeds up numerical solutions for general equilibrium models. Demand with aggregators also simplifies the analysis of industry equilibrium and interactions between firms under imperfect competition: firms under monopolistic competition take such aggregators as given, and firms under Bertrand competition account for how their choice influence such aggregators.[3]

[2]Our definition of rank and aggregators also differs from the notion of "latent separability" introduced by Blundell and Robin (2000). We focus on the rank of demand while Blundell and Robin (2000) relates more closely to the rank of income effects and homothetic aggregators used as part of utility and expenditure functions.

[3]Under Cournot, it may then be easier to work with inverse demand and express aggregators as functions of quantitites,

Beyond their applications to economics, these results can be useful for machine learning. Assuming a low rank is useful for various applications, such as principal component analysis, signal processing and matrix completion (e.g. "Netflix problem"), image compression, word embeddings and Latent Semantic Analysis (LSA), data imputation, etc. Proposition 1 characterizes any function with low-rank interactions between variables, while Proposition 3 can be used to construct homogeneous functions with a specific rank of the Hessian after removing diagonal terms.

1.1 Preamble: aggregators and the rank of cross-price effects

Suppose that we have J goods indexed by i or $j \leq J$, and that demand or any other function $F_i(x)$ corresponding good i depends specifically on its own price or a specific input x_i, and potentially also depends on all other prices or inputs $x_1, ..., x_J$. Cross-price effects can be of any rank up to J, in theory, even if we also impose assumptions such as rationality of a representative consumer. Hence, unless we impose ad hoc restrictions on empirical models, this implies that the number of cross-price elasticities to identify grows quadratically with the number of goods. As typical consumer data often includes more goods than markets or time periods (e.g. with barcode-level data), identification of fully-flexible cross-price effects becomes impossible.

A common approach in applied statistics and machine learning is to approximate these interactions by imposing a lower rank (any matrix can be approximated by a lower-rank matrix) and thereby reducing the dimensions of the estimation problem. Here, the low-rank approximation would only apply to part of the problem, as the overall rank of price effects (entire Jacobian of F) needs to remain full since demand for each good typically depends on its own price (non-zero diagonal terms). Thus, we propose to decompose price effects into a diagonal matrix of own-price effects (which has full rank yet only one coefficient by good) plus a low-rank matrix of cross-price effects.

This is achieved when demand for a good i depends on its own price x_i and a few scalar functions $\Lambda_k(x)$, $k = 1, ..., K$ with $K \leq J$, that summarize the effects of other prices. We call these functions "aggregators". If demand $F_i(x)$ for good i coincides with a function $W_i\big(x_i\,,\,\Lambda_1(x)\,,...,\,\Lambda_K(x)\big)$, we can see that the matrix of cross-price effects can be expressed as a product of two smaller matrices:

$$\frac{\partial F_i}{\partial x_j} \;=\; \sum_k \frac{\partial W_i}{\partial \Lambda_k}\frac{\partial \Lambda_k}{\partial x_j} \quad \text{for } i \neq j$$

Since the matrices with coefficients $\frac{\partial W_i}{\partial \Lambda_k}$ and $\frac{\partial \Lambda_k}{\partial x_j}$ have at most a rank K (they have K columns or rows), the resulting matrix of cross-price effects has at most a rank K, where K is the number of such aggregators.

In this section we show that the converse holds: if cross-price effects have a low rank K, the corresponding demand functions must depend on some K aggregator functions in addition to their own price.[4]

as discussed later in the paper.

[4] When own price effects are null, this result is known and easy to obtain. If we consider a J-dimensional function with rank-K Jacobian, its image corresponds to a K-dimensional embedded manifold. Here the difficulty arises from the

This result applies to smooth functions with rank and other regularity conditions, but we do not impose rationality of demand at this stage (we do not impose symmetry assumptions on the Jacobian). Specifically, the conditions that we impose relate to regularity and the rank of a non-diagonal component of the Jacobian:

C1. **Rank.** We assume that matrix $\frac{\partial F_i}{\partial x_j}$ is the sum of a diagonal matrix σ with diagonal elements σ_i and a matrix S of rank K.

C1'. **Identifying goods.** Furthermore, we assume that there are K goods $k = 1, ..., K$ such that the truncated matrix with coefficients $S_{kj} = \left\{\frac{\partial F_k}{\partial x_j} - \sigma_k \mathbb{1}(j = k)\right\}$, $k \leq K$, still has rank K.

C2. **Stability of the rank:**

i) The rank of the "substitution matrix" S remains K when we drop $K+1$ rows, corresponding to the price of all goods $k \leq K$ and any good $i > K$.

ii) The Jacobian of $(F_1(x), ..., F_K(x), x_1, ..., x_K, x_i, x_j)$ has full rank $2K + 2$, when we pick any two goods $i > K$ and $j > K$, $i \neq j$.

C3. **No escaping.** For any sequence $x(t)$ is such that $\max_{k \leq K} |F_k(x(t))| \to \infty$ while $x_1(t)...x_K(t)$ remains bounded, some other values of F are unbounded, i.e., $\max_{j>K} |F_j(x(t))| \to \infty$

C4. **Connectedness.** We assume that level sets of $(F_1(x), ..., F_K(x), x_1, ..., x_K)$ are connected, and remain also connected when we condition for the price of another good x_j.

The diagonal matrix σ captures own-price effects. Matrix S, which we refer to as the substitution matrix, then captures substitution patterns across goods, conditional on their own prices. Note that these assumptions on the stability of the rank imply that the matrix of cross-price effects has a "low" rank in a statistical sense; i.e., its rank K is smaller than $J/2$.

We obtain what is not yet our main theoretical result but a key motivation for our analysis of demand with price aggregators:

Proposition 1 *Under assumptions [C1], [C1'], [C2], [C3] and [C4], there exist K real functions $\Lambda_1(x)$, ... , $\Lambda_K(x)$ and J functions $W_i(\Lambda, x_i)$ such that:*

$$F_i(x) \;=\; W_i\Big(x_i\,,\,\Lambda_1(x)\,, ...,\,\Lambda_K(x)\Big)$$

i.e., demand for each good can be reduced to a function of its own price and K "aggregators" $\Lambda_k(x)$.

We relegate the proof to the Appendix, and here we provide a rough sketch.

In a first step, we show that for each good i the gradient of demand is a linear combination of the gradient of the demand for goods $k = 1$ to K, the gradient of x_1, ... , x_K (i.e., dummy indicators

presence of own-price effects, so that the Jacobian has full rank, and the image does not provide a simple way to define the K aggregators.

for each of these goods), as well as the price of its own good x_i. Applying Lemma 1 of Goldman and Uzawa (1956) Goldman and Uzawa (1964) and the connectedness assumption, this implies that we can write demand for goods $j > K$ as a function of its own price, the price of the first K goods, and the demand for the first K goods. In the next steps, the goal is to re-adjust demand for each good k by its own price in order to define each "aggregator" (so that we can drop each price x_k from the set of arguments).

To adjust each demand, we need to work in the $2K$ dimensional Euclidian space defined by demand and prices for the first K goods. In step 2, exploiting the assumptions on the stability of the rank, we obtain that the own-price elasticity σ_k of the first K goods can be expressed as a function of the prices and demands for the first K goods.

Using this property, in step 3, we provide autonomous differential equations that adjust demands by shifting their own prices by some arbitrary price changes. Thanks to assumption [C3], a global solution exists (instead of [C3], a sufficient condition is that own price elasticities σ_i are bounded). We show in step 4 that the functions obtained in step 1 are invariant to such adjustments. We show in step 5 that these adjustments can be made independently across all K goods, regardless of the ordering of these adjustments. Finally, we use this property in step 6 to define our aggregators (demands for goods 1 to K adjusted for their own prices), and show that demand for all goods depend only on these aggregators.

This proof puts a lot of importance on the first K goods, assumed to enter independently in matrix S of cross-price effects, as described in condition [C1']. This assumption is actually redundant and implied by [C1] locally, as we can always find K independent rows of a rank-K matrix. However here for practicality we impose that the same set of goods $1, ..., K$ can be used to find such independent rows. This is not innocuous as it allows us to define aggregators in a Euclidean space.

When we do not have a constant set of K goods with cross-price effects of rank K, the K aggregators are Euclidean only locally. Assuming that the rank assumptions hold only locally, i.e., dropping [C1'] and the constant set of goods, we can build on Proposition 1 to define aggregators on smaller subsets that are locally Euclidean. Then, by patching these subsets together, we obtain a manifold, which actually provides a more natural (but more abstract) mathematical environment for price aggregators:

Corollary 2 *Under local conditions [C1], [C2], [C3] and [C4], there exists a smooth map $\Lambda(x)$ to a K-dimensional manifold and some smooth function $W_i(x, \Lambda)$ such that: $F_i(x) = W_i\big(x_i\,, \Lambda(x)\big)$.*

In this section, note that we do not impose (yet) that demand is derived from rational behavior, and as such we do not rely on any symmetry assumption. The next results combine assumptions on aggregators and rationality, which implies some form of symmetry in cross-price effects.

2 Rational demand with price aggregators: homothetic case

We now examine the implications of rationality; i.e., that consumer demand is derived from utility maximization (under a standard budget constraint). We maintain the assumption that demand depends on its own price, as well as given number K of aggregators (themselves functions of all other prices). For the ease of exposition, in this section we start with the case of homothetic demand, and move to more general cases in the next section.

In the first subsection below, we precisely lay out hypotheses in addition to other topology assumptions, and in Proposition 3 we provide the functional forms of all demand functions that satisfy these properties. In addition, we provide a form of utility that yields such demand systems, assuming rational behavior under a standard budget constraint. This first set of results can be understood as a set of necessary conditions that demand must satisfy in order to depend on K aggregators and be derived from a utility function.

These necessary conditions however are not sufficient to ensure that such a utility function is quasi-concave (or, equivalently, obtained from a concave price index). We address this issue by providing (mild) additional sufficient conditions. We examine properties of the price substitution matrix in these cases.

These results provide practical ways to construct utility and demand with low-rank cross-price effects. As an illustration, we then provide a example of a very tractable form of demand that allows for flexible own price effects as well as cross-price effects parameterized by arbitrary semi-definite positive symmetric matrices, with a chosen rank K corresponding to the number of aggregators.

2.1 Set up and functional form in the homothetic case

Under homothetic preferences, demand q_i for each good $i \in \{1, ..., J\}$ is proportional to income w, hence expenditure shares $W_i \equiv \frac{p_i q_i}{w}$ only depend on prices p. More specifically, our goal here is to describe expenditure shares that depend on their own prices, respectively, and a vector of K price aggregators. This implies a rank K of cross-price effects, as discussed previously. We denote these aggregators by $\Lambda = (\Lambda_1, ..., \Lambda_K) \in \mathbb{R}^K$ (using subscript k to refer to an aggregator), Hence the expenditure share on good i is expressed as:

$$\frac{p_i q_i}{w} = W_i(p_i, \Lambda_1(p), ..., \Lambda_K(p)) \tag{1}$$

where p refers to the full vector of prices, and p_i is the price of good i. We assume that the number of goods is larger than the number of aggregators (specifically, $J > K + 3$), as our analysis will be particularly useful in settings where the number of goods is very large and where we want to reduce the dimensionality of interactions between goods. We consider only one period, with a balanced budget:

$$\sum_i W_i(p_i, \Lambda(p)) = 1 \tag{2}$$

Rationality We focus on demand from a rational consumer who is maximizing a quasi-concave utility U. This is equivalent to assuming the existence of a price index $P(p)$ that is homogeneous of degree one and concave in prices p. It is also the same as imposing that the expenditure function is proportional to $P(p)$ (i.e. multiplicatively separable in utility U and prices p). Shephard's Lemma implies that the expenditure share on good i must equal the derivative of $\log P$ w.r.t $\log p_i$, hence:

$$\frac{\partial \log P(p)}{\partial \log p_i} = W_i(p_i, \Lambda(p)) \tag{3}$$

For most of the analysis, we assume that demand (and utility) is smooth.

For the first proposition, we also make the following assumptions on price effects and topology:

Additional assumptions:

A1. Own price elasticity: *for each good, the own price effect is negative, i.e:* $\frac{\partial W_i}{\partial p_i}(p_i, \Lambda) < 0$*, holding* Λ *constant, evaluated at any* p *and* Λ*.*

A2. *For any* Λ *and any real* $y > 0$*, there exist a real* $t > 0$ *such that:* $\sum_i W_i(t, \Lambda) = y$*.*

A3. Rank of ∂W: *the matrix with coefficients* $\left\{\frac{\partial W_i}{\partial \Lambda_k}\right\}$ *has full rank* K*, where* K *denotes the number of aggregators.*

A4. Rank of $\partial \Lambda$: *the matrix with coefficients* $\left\{\frac{\partial \Lambda_k}{\partial \log p_i}\right\}$ *has maximal rank* K*, even if we drop one good* i *from the set of goods.*

A5. Connectedness. *The level sets of* Λ*,* $\left\{p \in \mathbb{R}^J \mid \Lambda(p) = \Lambda_0\right\}$*, are connected, for any* $\Lambda_0 \in \mathbb{R}^K$*.*

A6. No escaping: $\exists p \in \mathbb{R}^J$ *such that* $\max_i |\log W_i(p_i, \Lambda^{(t)})|$ *goes to infinity for any sequence of* $\Lambda^{(t)} \in \mathbb{R}^K$ *that escapes any compact set.*

Assumption 1 might be reversed, i.e. it may be possible to have all own price effects be positive, but we cannot have a combination of signs. The case of positive own price effects is considered for instance in Matsuyama and Ushchev (2017) for the special case of homothetic preferences with a single aggregator.

With the two rank assumptions (A3 and A4) we assume that aggregators capture different types of information relevant to consumers. With those, we basically assume that K is the minimum number of aggregators needed to explain demand. If the gradient of an aggregator was colinear with the gradient of other aggregators, we could then express this aggregator as a function of the other one, thereby reducing the number of aggregators.

Then, two topology assumptions put restrictions on the space of aggregators. The connectedness assumption A5 can be interpreted as a monotonicity assumption. It implies that if there are two sets of conditions in p that are associated with the same values of aggregators Λ, there is a continuous path indexed by $t \in [0, 1]$ of intermediate conditions $p(t)$ from one to the other with the same aggregator values $\Lambda(p(t)) = \Lambda$. On the contrary, topology assumption A6 implies that, when some aggregators diverge, there are some relative expenditures that also diverge (for some reference price level). Also, given the rank and topology assumptions, up to a change in variables, we can assume that $\Lambda(p)$ spans all $\mathbb{R}^K$.[5]

We will use Lemma 1 of Goldman and Uzawa (1964) which states that if the gradient of a real function f (defined on a Euclidean space) is colinear with the gradients of other real functions g_1, ..., g_n, and if the level sets of g are connected, we can express f as a function of $(g_1, ..., g_n)$. Assumption A5 on connectedness is useful for this.

We can now move onto the first main result on the functional form of homothetic demand:

Proposition 3 *Homothetic demand that depends on aggregators Λ and satisfies all assumptions A1-A6 above must take the form:*

$$W_j(p_j, \Lambda) \;=\; D_j(p_j/\Lambda^*, \Lambda') \tag{4}$$

where $\Lambda = \Phi(\Lambda^*, \Lambda')$ and Φ is a one-to-one re-mapping from aggregators $\Lambda \in \mathbb{R}^K$ to aggregators $H \in \mathbb{R}$ and $\Lambda' \in \left\{ \Lambda \;\middle|\; \sum_i W_i(1, \Lambda) = 1 \right\}$, a submanifold of dimension $K-1$. Moreover, it is derived from price index $P(p)$ that satisfies:

$$\log P(p) \;=\; \log \Lambda^* \;-\; G(\Lambda') \;+\; \sum_j \int_{t=1}^{p_j/\Lambda^*} D_j(t, \Lambda')\, d\log t \tag{5}$$

for some real function $G(\Lambda')$, and where the aggregators (Λ^*, Λ') are such that the partial derivatives of the RHS in (Λ^*, Λ') are null. Aggregator Λ^* is then homogeneous of degree one in p while Λ' is homogeneous of degree zero.

Not any function W_i of prices and aggregators Λ coincides with a rational demand system: a rational demand system with K aggregators Λ must take the form described above, with a utility characterized by (5). In particular, there must be one aggregator (denoted Λ^*) that plays a special role. We can think of this aggregator as adjusting in order to obtain a balanced budget: the first order condition in Λ^* in the RHS of (5) is equivalent to imposing $\sum_i D_i = 1$. In the special case of directly-additive preferences (see Section 5 and Fally 2022 for examples), aggregator Λ^* coincides with the budget multiplier. Regarding other aggregators Λ', note that: i) the good-specific demand function D_i can be a flexible function of Λ'; ii) aggregators Λ' are determined by the first-order condition (zero derivative of RHS) – we discuss in the next subsection the implications for price and income effects.

We relegate the proof of Proposition 3 to the Appendix but we provide here some intuition behind it. By examining the gradients and using Goldman and Uzawa's lemma, the first step is to show that V takes the following form: $\log P = -M(\Lambda) + \sum_i \int_1^{p_i} W_i(t, \Lambda) d\log t$. In addition, the rank of the gradients of Λ and $\log P$ imply that the derivatives of the right-hand side must be null for each

[5] A more natural yet abstract approach would be to define aggregators Λ as part of a smooth K-dimensional manifold. The proofs can be reformulated in this setting.

aggregator Λ_k (which we will refer to as the first-order condition in Λ_k). The envelope theorem then implies that the derivative of $\log P$ in $\log p_i$ equals $W_i(p_i, \Lambda)$, as desired. At this stage, however, nothing guarantees that such expenditure shares W_i sum up to one across goods.

The budget constraint (or, equivalently, homogeneity of P) further imposes functional form restrictions on the demand function. For such expenditure shares to add up to unity, it must be that the first-order condition in one of the aggregators (or a combination of those) implies the budget constraint. This is intuitively why one specific aggregator such as Λ^* in equations (4) and (5) plays a specific role. It must enter symmetrically (across goods) as a price shifter.

Differentiating the budget constraint and using the rank assumptions, we obtain that the own price effect $\frac{\partial W_i}{\partial \log p_i}$ must be colinear with the derivatives of W_i in Λ. Moreover, we obtain that coefficients of colinearity can be expressed as functions of Λ. We can then construct a "flow" Φ (transformation within the Λ space) that must keep each $W_i(tp_i, \Phi(t, \Lambda))$ invariant w.r.t t (assumptions 1, 2 and 6 are useful to define Φ globally). We use this flow to project the aggregators onto Λ^* and Λ'.[6]

The homogeneity of $\Lambda^*(p)$ (degree one) and Λ' (degree zero) is then simply obtained by checking that $\lambda\Lambda^*$ and Λ' are solutions of the first-order conditions when prices are multiplied by λ. Note that these come as a result as we have not initially imposed any homogeneity assumptions on the aggregators, aside from the assumption of homotheticity of preferences.

For the remainder of the paper, we denote the aggregators Λ^* and Λ, rather than Λ'.

2.2 Rationalization and concavity

Conversely, we can obtain sufficient conditions under which the P function defined above is concave in prices, which would then imply that there is a well-defined quasi-concave utility from which we can derive such demand systems. As for Proposition 3, we consider smooth functions $D_i(p_i, \Lambda)$ and $G(\Lambda)$ where $\frac{\partial D_i}{\partial p_i} < 0$ and define:

$$\log \widetilde{P}(p, \Lambda^*, \Lambda) \;=\; \log \Lambda^* \;-\; G(\Lambda) \;+\; \sum_j \int_{t=1}^{p_j/\Lambda^*} D_j(t, \Lambda)\, d\log t \tag{6}$$

This coincides with the price index when $\Lambda^*(p)$ and $\Lambda(p)$ are such that the derivative of the right-hand side is null in (Λ^*, Λ). The elasticity of P in each price p_i would then provide the expenditure shares described previously. What is left to provide are sufficient conditions for the concavity of P. This is obtained naturally by imposing concavity or convexity in Λ, conditions that are simpler to check.

Lemma 4 *Suppose that the function $\widetilde{P}(p, \Lambda^*, \Lambda)$ is defined as above.*

i) If $\log \widetilde{P}(p, \Lambda^*, \Lambda)$ is convex in Λ, define $P(p) = \max_{\Lambda^*}\{\min_\Lambda \widetilde{P}(p, \Lambda^*, \Lambda)\}$.

ii) If $\log \widetilde{P}(p, \Lambda^*, \Lambda)$ is concave in $(\Lambda, \log p)$, define $P(p) = \max_{\Lambda^*, \Lambda} \widetilde{P}(p, \Lambda^*, \Lambda)$.

In both cases, $P(p)$ is concave and homogeneous of degree one in p.

[6] For these steps is it helpful to consider the "Lyaponov" function, $\sum_i W_i(1, \Phi(t, \Lambda))$, that is strictly decreasing in t.

Conditional on aggregators Λ^* and Λ, it is easy to check that $\widetilde{P}(p, \Lambda^*, \Lambda)$ is concave in log prices. The max and/or min operations in Proposition 4 preserve concavity. First, in terms of Λ, this includes taking the maximum in Λ, noticing that the domain of Λ is assumed to be $\mathbb{R}^K$ and is convex, so the max of $\log \widetilde{P}(p, \Lambda^*, \Lambda)$ remains concave. Conversely, taking the minimum always preserves concavity, regardless of the domain of Λ.

By introducing Λ^*, we take the "perspective" of such function,[7] which allows us to go from log concavity to concavity in p, and also provides homogeneity in prices when we take the maximium over Λ^*. Perspective functions have been used recently in game theory, mean-field games, machine learning, transportation theory, among others (Combettes 2018, Combettes and Müller 2018).

A more general alternative is to assume that $\Lambda = (\Lambda^+, \Lambda^-)$ can be separated into two sets of aggregators, one set for which we have convexity and the other one where we have concavity. Our results carry by jointly taking the maxium over Λ^+ and the minimum over Λ^- if Slater conditions in $(\Lambda^+, \Lambda-)$ are satisfied.

When $P(p)$ is concave, demand can be alternatively derived from maximizing a concave utility function that is homogenous of degree one in quantities. Here, this utility function can be expressed with a similar functional form as P.

Proposition 5 *Under the assumptions of Lemma 4, the demand system can be obtained from the maximization (under the budget constraint) of the following utility function:*

$$\log U \;=\; -\log \Lambda^* \;-\; G(\Lambda) \;+\; \sum_i u_i(q_i \Lambda^*, \Lambda)$$

where each $u_i(q_i, \Lambda)$ is the Fenchel concave conjugate of $\int_{t=0}^{\log p_i} D_i(t, \Lambda)\, dt$ in p_i (conditional on Λ), and where we take the minimum/maximum over Λ depending on convexity/concavity.

To prove this result, it is useful to note that the log of utility is the Fenchel concave conjugate of the log of the price index (see Appendix). Fenchel concave or convex conjugates are used implicitly in various contexts in economics, for instance to retrieve cost and profit functions.[8] Here, in particular, each u_i is defined as:

$$u_i(q_i, \Lambda) \;=\; \min_{p_i} \left\{ p_i q_i - \int_{t=1}^{p_i} D_i(t, \Lambda)\, d\log t \right\}$$

and is concave in q_i.

Note that, for the optimal consumption basket as a function of prices, Λ^* and Λ in this formulation coincide with Λ^* and Λ in the dual.

[7] Take a real function $f(x)$ with $x \in \mathbb{R}^J$. For $t > 0$, the function $tf(x/t)$ is the perspective of f and is concave (resp. convex) in (x, t) if and only if f is concave (resp. convex) in x. See Combettes (2018) for properties of perspectives.

[8] The Fenchel conjugate of a convex function $f(x)$ is $f^*(y) = \max\{p.x - f(x)\}$, and instead we use the minimum for a concave function. The conjugate of the conjugate is equal to the original function (Fenchel-Moreau theorem).

2.3 Implications for price effects

When preferences are rational and allow for K aggregators, Theorem 1 states that one of such aggregators must play a special role and must enter symmetrically as a price shifter while demand can be a flexible function of other aggregators.

Furthermore, one should note that the first-order conditions in aggregators Λ imply some symmetry relating how Λ depends on prices p to how demand D depends on Λ itself. Differentiating the first-order conditions in Λ^* and Λ, we obtain that $\frac{\partial \Lambda_k}{\partial \log p_i}$ is tightly linked to $\frac{\partial D_i}{\partial \Lambda_k}$:

$$\frac{\partial D_j}{\partial \Lambda_k} \;=\; \Gamma_{k\Lambda^*}\frac{\partial \log \Lambda^*}{\partial \log p_j} + \sum_{k'} \Gamma_{kk'}\frac{\partial \Lambda_{k'}}{\partial \log p_j} \tag{7}$$

where matrix $\{-\Gamma_{kk'}\}$ corresponds to the Hessian of the right-hand-side of equation (5) in Λ and: $\Gamma_{\Lambda^* k} = \sum_j \frac{\partial D_j}{\partial \Lambda_k}$ (but note here that Γ is not constant and varies with prices). If the right-hand-side of equation (5) is convex in Λ, as in case i) of Lemma 4, matrix Γ is definite negative. In case ii) of Lemma 4, it is definite positive.

Using this relationship, we can illustrate more directly the rank of cross-price effects and the influence of aggregators. Holding aggregator Λ^* constant, we can exploit the J by K matrix $\frac{\partial W}{\partial \Lambda}$ and obtain the following expression for cross-price effects:

$$\left.\frac{\partial W_i}{\partial \log p_j}\right|_{\Lambda^*} \;=\; \sum_{k,k'} \gamma^S_{kk'}\,\frac{\partial D_j}{\partial \Lambda_{k'}}\frac{\partial D_i}{\partial \Lambda_k} \tag{8}$$

where $\gamma^S_{kk'}$ is the inverse of the $\Gamma_{kk'}$ matrix above restricted to the first $K-1$ entries excluding Λ^* (Hessian of $\widetilde{P}$ in Λ), and has a rank $K-1$. This matrix of cross-price effects inherits properties of Γ (e.g., it is positive semi-definite if Γ is positive definite) but its cells do not necessarily have same sign. In either case, it allows for complementary between some of the goods, unlike additive random utility models (ARUM) such as mixed logit.

2.4 Example of a semi-parametric specification

In this section, we illustrate the usefulness of these results by exploring a specification based on a linear relationship between own prices and aggregators. Special cases of it includes nested logit/CES and the more recent Inverse Product Differentiation Logit model (IPDL, Fosgerau et al 2024).

Suppose that the price index is:

$$\log P \;=\; \max_{\Lambda^*,\Lambda}\Big\{ -\log\Lambda^* \;-\; \sum_k g_k(\Lambda_k) \;-\; \sum_i S_i\Big(\log p_i + \log\Lambda^* + \sum_k b_{ik}\Lambda_k\Big)\Big\}$$

with $S_i(x_i) = \int_{x_i}^{+\infty} D_i(t)dt$ and $D_i' < 0$, where $b_{ik} \in \mathbb{R}$ are parameters and where g_k are convex functions with $g_k' > 0$ and $g_k'' > 0$. Also, suppose that aggregators are such that the derivative of V

in Λ^* and Λ is null. The right-hand side is concave in $\log p$, $\log \Lambda^*$ and Λ, so aggregators are uniquely identified and the resulting demand system is well defined, as described in Lemma 4.

Each good can have a flexible own-price demand schedule D_i, as long as it is downward slopping.

The first-order condition in Λ^* yields the adding-up condition: $\sum_i D_i = 1$, so we obtain that the expenditure share on good i is given by each of these terms, with $W_i \;=\; D_i\Big(\log p_i + \log \Lambda^* + \sum_k b_{ik}\Lambda_k\Big)$. In turn, the first-order condition for each aggregator Λ_k yields a simple expression as a sum of the b_{ik}'s (for each k) weighted by expenditure shares:

$$g'(\Lambda_k) \;=\; \sum_i b_{ik} D_i\Big(\log p_i + \log \Lambda^* + \sum_{k'} b_{ik'}\Lambda_{k'}\Big) \;=\; \sum_i b_{ik} W_i \tag{9}$$

Special case. Several specifications of functions g_k lead to very tractable solutions. For instance, as we describe later, nested logit/CES and IPDL (Fosgerau et al 2024) can be obtained by chosing exponential for each g_k and iso-elastic functions D_i. Here we explore an even simpler case by imposing a quadrative separable specification for the g's. The expression for Λ_k is the simplest with $g_k = \frac{1}{2}\Lambda_k^2$, so that:

$$\Lambda_k \;=\; \sum_i b_{ik} W_i$$

In that case, we can also express expenditure shares more directly as:

$$W_i \;=\; D_i\Big(\log p_i + \log \Lambda^* + \sum_j \widetilde{\Gamma}_{ij} W_j\Big) \tag{10}$$

where $\widetilde{\Gamma}$ is a semi-definite positive symmetric matrix with coefficients $\widetilde{\Gamma}_{ij} = \sum_k b_{ik} b_{jk}$.[9] This matrix is constant (does not vary with prices) and depend only on the primitive parameters b_{ik} to estimate. Aside from the canonical aggregator Λ^*, all cross-price effects are captured by the term $\sum_j \widetilde{\Gamma}_{ij} W_j$ and yield:

$$\left.\frac{\partial W_i}{\partial \log p_j}\right|_{\Lambda^*} \;=\; \sum_k D_i' \, b_{ik} \frac{\partial \Lambda_k}{\partial \log p_j} \;=\; \sum_{j'} D_i' \, \widetilde{\Gamma}_{ij'} \; \left.\frac{\partial W_{j'}}{\partial \log p_j}\right|_{\Lambda^*} \tag{11}$$

for $i \neq j$ (holding Λ^* constant). In this expression, we can see that the effect of p_j on expenditures W_i on i can be expressed as a simple linear combination of how this affect expenditures on other goods, especially its own good W_j.

Using the results in the previous section (expression 8 above), we can further reduce it to:

$$\left.\frac{\partial W_i}{\partial \log p_j}\right|_{\Lambda^*} \;=\; D_i' \, D_j' \sum_{k,k'} \gamma^S_{kk'} \, b_{ik} \, b_{jk'} \tag{12}$$

where γ^S is the inverse of the matrix with coefficients $\mathbb{1}_{(k=k')} - \sum_i b_{ik} b_{ik'} D_i'$. This demand system is very flexible already as it can fit a wide range of substitution patterns. To be more precise, conditional

[9] We can also simply check homotheticity by shifting all prices (in log) by a common constant term. This will increase $\log \Lambda^*$ and $\log P$ by that same constant term, without affecting expenditure shares and other aggregators, hence $P(p)$ is homogenous of degree 1 in prices.

on own-price effects $\{D_i'\}$, we can choose parameters b to fit any rank-K semi-definite positive matrix γ^S with a spectral radius less than one, i.e. that does not have eigenvalues above one (see Appendix for a proof).

In this specification, recovering welfare is not difficult. Once own demand and cross-price effects are estimated, we can recover $S_i(x_i) = \int_{x_i}^{+\infty} D_i(t)dt$ by integrating. In turn, we can recover $\frac{1}{2}\sum_k \Lambda_k^2 = \frac{1}{2}\sum_{i,j} \widetilde{\Gamma}_{ij} W_i W_j$ as the quadratic form associated with Γ and evaluated using expenditure shares.

Note also that such demand can be derived from the following direct utility:

$$\log U \ = \ \log \Lambda^* \ + \ \frac{1}{2}\sum_k \Lambda_k^2 \ + \ \sum_i u_i \Big(\log q_i - \log \Lambda^* - \sum_k b_{ik}\Lambda_k \Big)$$

where aggregators are now defined as a funciton of q, and such that the right-hand-side has a zero derivative in $\Lambda^*(q)$ and $\Lambda(q)$.

3 Non-homothetic demand with price aggregators

3.1 Utility as an additional aggregator

In the previous results, expenditure shares are assumed to be derived from a price index that is homogeneous in prices. Under non-homotheticity, we can apply the same results to the expenditure function, conditional on utility. We would then be considering demand where expenditure shares are function

$$\frac{p_i q_i}{w} \ = \ W_i(p_i, \Lambda_1(p,U), ..., \Lambda_K(p,U), U) \tag{13}$$

The rationality condition is then expressed using the expenditure function $e(p,U)$ which must have the same properties as the price index in p (homogeneity and concavity) but may also depend on utility. Specifically, Shephard's Lemma requires:

$$\frac{\partial \log e(p,U)}{\partial \log p_i} \ = \ W_i(p_i, \Lambda(p,U), U) \tag{14}$$

Rank and topological conditions [A1]-[A6] remain identical in terms of prices and aggregators. Under these conditions, using Proposition 3, we obtain the following functional forms for demand and the expenditure function:

Corollary 6 *Demand that depends on aggregators (Λ, U) and satisfies all assumptions A1-A6 above must take the form:*

$$W_j \ = \ D_j(p_j/\Lambda^*, \Lambda, U) \tag{15}$$

Moreover, it is derived from an expenditure function $e(p,U)$ that satisfies:

$$\log e(p,U) \ \ = \ \ \log \Lambda^* \ - \ G(\Lambda, U) \ + \ \sum_j \int_{t=1}^{p_j/\Lambda^*} D_j(t, \Lambda, U)\, d\log t \tag{16}$$

for some real function $G(\Lambda, U)$, and where the aggregators (Λ^*, Λ) are such that the partial derivatives of the RHS in (Λ^*, Λ) are null.

Conversely, if the right-hand side satisfies the concavity or convexity conditions highlighted in Lemma 4, the corresponding expenditure function is concave and homogeneous of degree one in prices, hence can be associated with rational preferences.

In this setting, one can think of utility as an additional aggregator, and replace U by indirect utility if we want to express demand as a function of prices and income. Aggregators would then depend on income (through indirect utility) and would be homogeneous of degree zero jointly in income and prices.

Example. These results can be readily applied to the specification discussed in the section above. Perhaps the most simple way to obtain flexible good-specific Engel curves is to incorporate an additive shifter $\alpha_i(U)$ that is itself a function of utility. Expenditure shares, expressed as Hicksian demand, would then be:

$$W_i \;=\; D_i\Big(\alpha_i(U) + \log p_i - \log \Lambda^* + \sum_k b_{ik}\Lambda_k\Big)$$

while keeping the same expressions for aggregators Λ_k as in equation (9).

3.2 Non-homothetic case with an explicit expression for V

For tractability and to more easily recover indirect utility from prices and observed demand patterns, an alternative approach is to extend Proposition 3 by working directly as a function of prices and income instead of prices and utility. In the results below, we obtain an expression for indirect utility, which in some cases may be more readily applied to consumer welfare analysis.

In this setting, it is useful to consider all objects as functions of the log of normalized income, $x_i = \log(p_i/w)$, for each good i, and express everything in terms of x instead of p. We delegate this analysis to the Appendix E. In brief, we find very similar functional forms as in Proposition 3, yet in terms of normalized prices and with flexible functions of the special aggregator Λ^*.

4 Relationship to specifications from the literature

Here we describe the link to previous forms of separability and demand systems studied in the literature, and discuss how our approach can be used to adopt a more general yet tractable approach to modeling demand. We start by summarizing demand with a single aggregators (Fally, 2022) which lead to demand with cross-price substitution matrix of rank one, then move onto standard approaches to modeling more complex cross-price effects, including i) EASI/AIDS, ii) combining multiple industries, iii) aggregating heterogeneous consumers as in Berry et al (1995). We then provide a general specification that aims to capture the best features of all these approaches while remaining tractable and amenable to estimation.

4.1 Single and double aggregator demand

Fally (2022) examines demand with two aggregators when one of the aggregators is utility U, but without imposing homogeneity in aggregator Λ^*. The special case where Λ^* is homogeneous, and thus just a function of prices, is then a special case of Corollary (6) and leads to a convenient expression for the expenditure function that facilitates welfare analysis. Demand then takes the form:

$$W_i \;=\; D_i\left(p_i/\Lambda^*\,,\, U\right)$$

and must be obtained from the following expenditure function:

$$\log e(p,U) \;=\; \max_{\Lambda^*}\Big\{\log \Lambda^* + \sum_j \int_{t=0}^{\log(p_j/\Lambda^*)} D_j(t,U)\,dt\Big\}$$

As long as good-specific demand functions D_j are decreasing in its first argument, this expenditure function $e(p,U)$ is concave and homogeneous of degree one in prices. In addition, one must also ensure that it is increasing with utility U.

While cross-price effects are then very simple under this specification (rank 1 or 2), the two-aggregator case is already flexible enough to construct demand systems with flexible own-price effects as well as flexible Engel curves. This shows by the negative that it is not necessary to consider more than two aggregators if we are not concerned with modeling interesting cross-price effects.

4.2 EASI and AIDS with aggregators

A well-known and simple demand system that allows for flexible cross-price effects is the EASI demand system developed by Lewbel and Pendakur (2009), which can be seen as a generalization of the AIDS by Deaton and Muellbauer (1980).

Here we can obtain EASI preferences as a special case of our demand systems by considering the following expenditure function:

$$\log e(p,U) \;=\; \log \Lambda^* \;-\; \frac{1}{2\gamma_k}\sum_k \Lambda_k^2 \;+\; \sum_i \int_0^{\log(p_i/\Lambda^*)} D_i\big(t,\Lambda,U\big)\,dt \qquad (17)$$

with linear demand function D_i:

$$D_i\big(t,\Lambda,U\big) \;=\; \alpha_i(U) \;-\; \theta_i\, t \;-\; \sum_k b_{ik}\Lambda_k \qquad (18)$$

where we impose $\sum_i \alpha_i(U) = 1$ for all levels of utility U, as well as $\sum_i b_{ik} = 0$, and where Λ and Λ^* are such that the RHS of (17) has zero derivatives in Λ and Λ^* (taking the maximum or minimum depending on the sign of γ_k).

Solving for Λ and Λ^* in this maximiation (see details in appendix), we obtain the following expenditure shares on good i with log-linear price effects and flexible Engel curves for each good (dictated by α_i):[10]

$$W_i \;=\; \alpha_i(U) \;+\; \sum_j \beta_{ij} \log p_j$$

where the coefficients $\beta_{ij} \;=\; \left[-\theta_i \mathbb{1}(i=j) + \frac{\theta_i \theta_j}{\sum_{j'} \theta_{j'}}\right] \;-\; \sum_k \gamma_k b_{ik} b_{jk}$, which satisfy the standard conditions imposed with EASI, i.e. $\sum_j \beta_{ij} = 0$ and $\beta_{ij} = \beta_{ji}$.

Conversely, note that any symmetric matrix β with column sums equal to zero can be decompose in this manner. Adjusting for own-price effects, the rank of β determines the number of aggregators K that are needed. Eckart-Young-Mirsky theorem then states that the largest eigenvalues of β (net of own price effects) would determine the quality of a lower-rank approximation.

4.3 Mixed logit/CES

Another common way to model non-trivial cross-price effects is to assume that a market is the aggregation of heterogeneous consumers (e.g. Berry 1994, Berry et al. 1995). Each consumer has Logit or CES preferences, but heterogeneous price elasticities and heterogeneous demand shifters (often modeled as heterogeneous evaluations of various product attributes). Two products have a greater cross-price effects (i.e. are more substitutes) if they tend to be purchased by the same types of consumers. At the aggregate level, we show here that we can interpret such mixed logit demand as demand with price aggregators, where we have at most one aggregator by consumer type. Hence, a more complex demand system with a larger number of consumer types leads to a greater number of aggregators and a higher rank of cross-price effects.

Formally, suppose that demand is the aggregation of several types of consumers, indexed by k, each of which has an expenditure share given by a multinomial logit structure as standard in discrete-choice models.[11] Expenditure shares for type k of consumers are then given by:

$$\widetilde{W}_{ik} \;=\; \frac{e^{-\alpha_k \log p_i + b_{ik}}}{\sum_j e^{-\alpha_k \log p_j + b_{ik}}} \tag{19}$$

Denote by ω_k the aggregate income share of consumers of type k. The aggregate expenditure share is then:

$$W_i = \sum_k \omega_k \widetilde{W}_{ik} \;=\; \sum_k \omega_k \Lambda_k e^{-\alpha_k \log p_i + b_{ik}} \tag{20}$$

with $\Lambda_k \;=\; \left(\sum_j e^{-\alpha_k \log p_j + b_{ik}}\right)^{-1}$. This specification coincides with a special case of the demand

[10] Here for the sake of exposition we omit the interaction terms between price effects and utility. Such interactions can be obtained by adding a term $\sum_l \gamma_{il} \Psi_l$ with coefficients satisfying $\sum_i \gamma_{ik} = 0$, combined with additional aggregators Ψ_l.

[11] As standard in the literature, we can assume that goods i differ in terms of their characteristics Λ^*, with ζ_{ih} describing the content of good i in characteristics Λ^*. Suppose that each type k of consumers has a valuation β_{kh} of characteristics Λ^*, we could have then: $\log b_{ik} = \sum_h \beta_{kh} \zeta_{ih}$.

described above, where the price index is defined as:

$$\log P = \max_{\Lambda^*,\Lambda} \left\{ -\log \Lambda^* - \sum_i \sum_k \Lambda_k b_{ik} e^{-\alpha_k(\log p_i + \log \Lambda^*)} + \gamma_0 \Pi_k \left(\Lambda_k\right)^{\gamma\omega_k/\alpha_k} \right\} \tag{21}$$

4.4 Nests and IPDL (Fosgerau et al 2024)

Consider different partitions of the set of goods. For instance, among yogurt, one partition could be composed of the set of vanilla-flavored yogurt, the set of plain yogurt, and the set with other favors, so that each yogurt product is in either one of these sets. On top of this, we can consider another partition depending on the fat content, or quality labels, etc.

A standard way to model partition is nested logit, but this only allows for one partition, with more or less fine sets within that partition. A recent paper by Fosgerau, Monardo, and De Palma (2024) provides a useful way to consider several of such partition at once, with a different substitution parameter for each partition, which they call it IPDL (inverse product differentiation logit model).

This can be viewed as a special case of our preferences, because demand can be expressed in terms of its own price, as well as an aggregator for each set of each partitition. Here below we provide an intermediate generalization where preferences remain homothetic (using log prices instead of price levels in IPDL) and where we can pick any functional form of demand for own-price effects instead of the logit/CES formulation.

Denote each partition by $\mathcal{P}$. and by $S \in \mathcal{P}$ the sets in that partition. For each of such set, we define an aggregator $\Lambda_{S,\mathcal{P}}$. We show here that this is a special case of the specification highlighted in Section 2.4 where the G function is the exponential and where we specify $b_{i,\mathcal{P},S} = \mu_{\mathcal{P}} \mathbb{1}_{(i\in S)}$, i.e. the inclusion function for set S and a parameter $\mu_{\mathcal{P}}$ that will capture substitution within vs. across goods of sets $S \in \mathcal{P}$.

Suppose that the price index is:

$$\log P \ = \ \max_{\Lambda^*,\Lambda} \Big\{ \log \Lambda^* \ - \ \sum_{\mathcal{P}} \sum_{S\in\mathcal{P}} \exp(\Lambda_{S,\mathcal{P}}) \ - \ \sum_i S_i \Big(\log(p_i/\Lambda^*) + \sum_{\mathcal{P}} \sum_{S\in\mathcal{P}} \mathbb{1}_{(i\in S)} \mu_{\mathcal{P}} \Lambda_{S,\mathcal{P}} \Big) \Big\}$$

The RHS is concave in the $\Lambda_{S,\mathcal{P}}$'s, so we can readily apply Lemma 4 and Proposition 5. The first-order conditions yield simple expressions: $\Lambda_{S,\mathcal{P}} = \log W_{S,\mathcal{P}}$, where $W_{S,\mathcal{P}} = \sum_{i\in S} W_i$ is the aggregate expenditure share on goods i in the set S (for that partition $\mathcal{P}$). Incorporating into the expression for the optimal consumption basket, the resulting expenditure share on good j is then:

$$W_j \ = \ D_j \Big(\log(p_j/\Lambda^*) \ - \ \sum_{\mathcal{P}} \sum_{S\in\mathcal{P}} \mathbb{1}_{(i\in S)} \mu_{\mathcal{P}} \log W_S \Big)$$

As with IPDL, the terms $\mu_{\mathcal{P}} \log W_S$ account for different substitution patterns with goods that are in the same set, allowing for i) different layers or partitions $\mathcal{P}$; ii) different intensity of substitution parameters $\mu_{\mathcal{P}}$ across partitions. Note again that these partitions overlap perfectly, in the sense that each good is in exactly one set for each partition.

In particular, this provides one way to classify goods into product categories. Below we provide a different approach, allowing for potentially more heterogeneous types of preferences across groups of goods (or industries).

4.5 Combining sectors

For this application, consider k as indexing industries, i.e. sets of goods, with K industries forming a partition of the full set of goods. Some simple assumptions on separability (e.g. convex combinations of indirectly-separable preferences) can lead to demand satisfying the functional form in equation (1). Here we describe how to relate these cases to the general formulation of Theorem 2 using duality and the Legendre-Fenchel transformation.

Combining indirectly-additive preferences First, consider a multi-tiered combination of indirectly-additive preferences across sectors k:

$$V = g(v_1(x_1), ..., v_K(x_K)) \tag{22}$$

where g is a strictly convex function from a convex subset $X \subset \mathbb{R}^K$ into $\mathbb{R}$, and where each v_k is a convex function of a vector x_k of log normalized prices $\{x_i\}_{i \in k}$ for the subset of goods categorized in sector k.

Denote by G the convex conjugate of g, i.e.:

$$G(\Lambda) \;=\; \max_v \big\{ -g(v) + \sum_k \Lambda_k v_k \big\} \tag{23}$$

Using the fact that the conjugate of the conjugate is the initial function (Fenchel-Moreau Theorem), we obtain:

$$V = g(v_1(x_1), ..., v_K(x_K)) \;=\; \max_\Lambda \big\{ -G(\Lambda) + \sum_k \Lambda_k v_k(x_k) \big\} \tag{24}$$

with Λ varying across all values taken by the gradient of g. In that case, the aggregators $\Lambda(x)$ evaluated at x coincides with the gradient of the upper-layer function g:

$$\Lambda(x) = \frac{\partial g}{\partial v_k}(v_1(x_1), ..., v_K(x_K)) \tag{25}$$

This equality comes from the first-order condition of equation (23).

Generalizations We can further generalize the insight above by transforming a combination of industries with demand with aggregators in each industry into a form that is a special case of Proposition 3. If demand in each industry k has m_k aggregators, then overall demand has $K = \sum_k m_k + 1$ aggregators at most. Details are provided in the Appendix, where we also describe the case of directly-additive preferences combined across multiple sectors.

5 Estimation approach

Here we are interested in situations with many goods (large J) where it is not practically possible to estimate fully flexible cross-price effects of the order of J^2.

Demand with aggregators provides a flexible framework to modeling and estimating demand with non-trivial cross-price effects[12] when the number of goods is large and we need to reduce the rank of cross-price substitution matrix. Our theoretical results impose discipline on the structure of these effects, exploiting the mathematical structure of demands for rational utility-maximizing consumers.

Our main specifications follows Section 2.4. To keep the focus on cross-price effects, we shut down income effects by developing a demand system which assumes homothetic preferences. In a set of extensions we then explore different ways to account for more flexible income effects.

5.1 Data

The Nielsen home scanner data[13] are collected through hand-held scanner devices that households use at home after their shopping in order to scan each individual transaction they have made. The data cover a quarter billion dollars of grocery expenditures, from about 60,000 individual households spread evenly across 53 "Scantrack" markets in the US (which approximately coincide with large metropolitan areas). Here as an illustration we focus on yogurt products, and further narrow the scope of our analysis on the 2010–2019 time period.

For this application, we aggregate expenditure by market at the monthly level. Hence, our dataset is similar to those typically used in the Industrial Organization literature, following Berry et al (1995), estimating demand across granular products and markets (or stores). We augment our dataset using information on product attributes from Label Insight, which we can match for about 80% of observed expenditures (excluding generic store brands, non-barcode and over the counter products).

5.2 Specification: additional functional form assumptions

As indicated above, we adopt a homothetic specification so as to focus on cross-price effects. Our starting point is the specification highlighted above (equation 10 in section 2.4) where aggregators enter linearly in combination with the own price of the good. We also assume that variation across markets and products is influenced by unobserved taste shocks ε_{it}.

Inverting the own demand curve D_i for each good i, we obtain the following specification for each market/time t:

$$D_i^{-1}(W_{it}) = \log p_{it} - \log \Lambda_t^* + \sum_j \widetilde{\Gamma}_{ij} W_{jt} + \varepsilon_{it} \tag{26}$$

Intuitively, own price effects are influenced by the shape of D_i while cross-price effects are determined

[12] Breaking away from the independence of irrelevant alternatives, "IIA", as would be implied by CES or Logit models

[13] Disclaimer: The conclusions drawn from the Nielsen data are those of the researcher(s) and do not reflect the views of Nielsen. Nielsen is not responsible for, had no role in, and was not involved in analyzing and preparing the results reported herein.

by the rank-K matrix $\widetilde{\Gamma}$ with coefficients $\widetilde{\Gamma}_{ij} = \sum_k b_{ik} b_{jk}$. Both have to be estimated. Aggregator Λ_t^* can be regarded as a price index or market fixed effect that is uniform across all goods.

In all that follows, we interpret the error term as idiosyncratic demand shock at the market t by good j level, which we treat as an additive price shifter in logs. Notice that, after inverting the left-hand side, there is no restriction on the domain of ε_{it}.

Parameterization of D. Various parameterizations would be convenient for D_i; all that is required is that it be strictly decreasing. A first specification that we favor is iso-elastic, parameterized by a product-specific shifter α_i and potentially product-specific elasticity θ_i:

$$D_i(t) = \exp\left[-\alpha_i - \theta_i t\right] \tag{27}$$

so that its inverse $D_i^{-1}(W_{it}) = -\frac{1}{\theta_i}(\log W_i + \alpha_i)$ leads to the simple log-linear specification

$$\log W_{it} = -\alpha_i - \theta_i \log p_{it} + \theta_i \log \Lambda_t^* - \theta_i \sum_j \widetilde{\Gamma}_{ij} W_{jt} - \theta_i \varepsilon_{it}.$$

We can also consider a parameterization of D_i that allows for a choke price, i.e. such that demand is null above a certain reservation price. We then consider:

$$D_i(t) = \nu \left[e^{-\alpha_i - \theta_i t} - 1\right] \tag{28}$$

so that demand for good i is positive if and only if the right hand side of expression (26) is smaller than $-\alpha_i$.

Projecting on attributes: our baseline estimation equation Following BLP and related approaches, a natural assumption is that goods with similar characteristics are better substitutes than others. Here, a simple way to capture this idea is to project the good/aggregator-specific demand shifter b_{ik} onto the space of product characteristics. Specifically, we use data ζ_{il} on attributes l across goods i, informing on whether good i has attribute l (in which case ζ_{il} is a dummy variable) or the intensity of that attribute (ζ_{il} is then a scalar). Using such data, we now impose:

$$b_{ik} = \sum_l \beta_{kl} \zeta_{il}$$

With such specification, cross-price effects between goods i and j are then fully determined by how i and j differ in terms of their characteristics. Aggregators are now determined by expenditure shares across attributes:

$$\Lambda_{kt} = \sum_l \beta_{kl} Z_{lt},$$

where Z_{lt} is defined as the sum of expenditure shares W_{it} across goods i weighted by their observed attribute ζ_{il}:

$$Z_{lt}(\zeta, W) \;=\; \sum_i \zeta_{il} W_{it}.$$

When ζ_{il} is a dummy (e.g., whether yogurt i has a vanilla flavor), Z_{lt} is simply the observed share of expenditures across products that have such a label. In turn, cross-price effects are determined by an interaction between observed characteristics ζ_{il}, weighted expenditures $Z_{l't}$ on characteristics l', and matrix $\overline{\Gamma}_{ll'} = \sum_k \beta_{kl}\beta_{kl'}$ capturing substitution patterns between attributes l and l'. Matrix $\overline{\Gamma}$ now becomes the key positive semi-definite matrix to be estimated.

With this projection on attributes, the estimating equation can be written as:

$$D_i^{-1}(W_{it}) = \log p_{it} - \log \Lambda_t^* + \sum_{l,l'} \overline{\Gamma}_{ll'}\, \zeta_{il} Z_{l't} + \varepsilon_{it}. \tag{29}$$

Relation to Berry et al (1995)) demand inversion. A major advantage of specification (29) is that cross-price effects enter linearly and do not enter the inversion of own expenditure shares on the left-hand side. In contrast, the inversion in BLP (Berry, Levinsohn and Pakes, 1996) relies on the mixing parameters that both characterize the heterogeneity of tastes across consumers and the cross-price effects, and is highly non-linear in these parameters. With BLP, one must therefore invert again to re-evaluate the orthogonality condition for each new set of heterogeneity parameters. Here, the inversion can rely on analytical solutions, and the linear specification in $\widetilde{\Gamma}$ and $\overline{\Gamma}$ leads to more transparent identification and helps to avoid the problem of weak instruments highlighted in Gandhi and Houde (2019).

5.3 IV and GMM formulation

We propose two alternative estimation strategies: i) a linear IV estimator imposing $\overline{\Gamma}$ to be symmetric but not necessarily positive semi-definite; ii) a GMM specification of a non-linear estimator imposing both symmetry and positive semi-definiteness on $\overline{\Gamma}$. In both cases, we use a set of instruments based on Gandi and Houde (2019), which can readily apply to our framework.

5.3.1 Linear Problem

Requiring the estimated matrix $\hat{\overline{\Gamma}}$ to be positive semi-definite involves a set of non-linear constraints (or alternatively, makes the objective function nonlinear in parameters). But if we set aside the requirement that $\overline{\Gamma}$ must be positive semi-definite, then we can express the estimation problem as a linear problem.

Starting from (29), combined with the log-linear specification of D_i and a common θ, we obtain the linear expression:

$$-\log W_{it} = \alpha_i + H_t + \theta \log p_{it} + \sum_{l,l'} \theta \overline{\Gamma}_{ll'} \zeta_{il} Z_{l't} + \epsilon_{it}, \tag{30}$$

where $H_t = \theta \log \Lambda_t^*$.

Observed variables are $\log W_{it}$, $\log p_{it}$ and $\zeta_{il} Z_{l't}$, with $Z_{l't} = \sum_i \zeta_{il'} W_{it}$. Linear coefficients to be estimated are $\theta \overline{\Gamma}_{ll'}$ and θ, as well as α_i and H_t which can be interpreted respectively as product and time/market fixed effects or nuisance parameters.

Further, as requiring $\Gamma_{ij} = \Gamma_{ji}$ for $i \neq j$ is a linear set of constraints, we can impose symmetry on $\overline{\Gamma}$ without compromising the linearity of the estimator. This leaves us with the need to estimate $1 + M(M+1)/2$ parameters, and so a need for at least this many restrictions.

Suppose that we have appropriate instruments, both valid and sufficiently numerous to achieve identification (see discussion of the differentiation instruments below). Still, note that estimating this equation using conventional linear estimators will involve estimating $1 + M^2$ parameters (not counting the nuisance parameters α and H). This is more than the available linear independent differentiation instruments ($1 + M(M+1)/2$). As a formal matter symmetry implies $M(M-1)/2$ linear restrictions, so that the estimator is identified. However, textbook implementations of, say, two-stage least squares typically assume identification even /absent/ additional linear restrictions (Greene and Seaks 1991 note a similar issue for the case of restricted OLS).

Do note, however, the resulting linear estimates $\overline{\Gamma}_{LIV}$ may be defective, as there is no guarantee that the linear estimates of $\overline{\Gamma}$ will yield a positive semi-definite matrix. We discuss an approach to correcting this using a non-linear GMM estimator below.

Differentiation Instrumental Variables Gandhi and Houde (2019) consider the problem of estimating a Berry, Levinsohn, and Pakes (1995) style model of differentiated products. This is *not* our model, but is a non-linear GMM problem using aggregate data that poses issues similar to ours as the number of characteristics grows large. They are interested in particular with the problem of weak instruments, and argue (formally in the context of the BLP model, but heuristic arguments may also apply to our setting) that the appropriate instrument set should include what they call "differentiation IVs," which capture differences in characteristics across goods.

In our case, characteristics for J different products are encoded in a $J \times M$ matrix ζ. Thus, differences across products for characteristic m can be written as

$$d_m = [\zeta_{jm} - \zeta_{j',m}]$$

yielding a $J \times J$ matrix of differences. Similarly, $d_p^s = [\log p_{js} - \log p_{j',s}]$ gives differences in log prices across goods for a market-periods s. (Note the important difference that d_m is invariant across s.)

Importantly, not all goods are available in all market-periods. Let A be an $S \times J$ "availability" matrix of ones and zeros. We can then construct a symmetric measure of difference for characteristic m using Ad_m^2, yielding an $S \times J$ instrument matrix. Extending this idea, what Gandhi and Houde

(2019) call the “instrument function” can be written as

$$G(\mathbf{p}, \zeta) = \begin{cases} A\, d_p^2 & \text{Differentiation in price;} \\ A\, d_m^2 & \text{Differentiation in } \zeta_m \text{ for } m = 1, \ldots, M \\ A(d_m \odot d_\ell) & \text{Interaction between differences in characteristics } \ell \text{ and } m, \end{cases} \tag{31}$$

where $\odot$ is the Hadamard product. This gives us $1+(M+1)M/2$ instruments to exploit, and identifies the linear specification described above as well as the GMM approach that we will now describe.

5.3.2 Generalized Method of Moments

To guarantee that the estimated matrix $\overline{\Gamma}$ is positive semi-definite we must turn to a non-linear estimator. Here we are concerned with the problem of estimating a $M \times M$ matrix $\overline{\Gamma}$, restricting $\overline{\Gamma}$ to be symmetric, positive semi-definite, and of rank $K \leq M$. A simple way to guarantee that $\overline{\Gamma}$ satisfies these restrictions is to instead choose a $M \times K$ matrix $\boldsymbol{B}$, such that $\boldsymbol{B}\boldsymbol{B}^\top = \theta\overline{\Gamma}$. This gives the required structure for $\theta\overline{\Gamma}$, but does not uniquely determine $\boldsymbol{B}$, as any matrix $\boldsymbol{B}^* = \boldsymbol{B}\boldsymbol{R}$ with $\boldsymbol{R}$ any orthonormal matrix would also have $\boldsymbol{B}^*\boldsymbol{B}^{*\top} = \theta\overline{\Gamma}$. However, if we require $\boldsymbol{B}$ itself to be an orthogonal matrix, so that $\boldsymbol{B}^\top\boldsymbol{B}$ is diagonal, then for any positive semi-definite $\overline{\Gamma}$ with rank less than or equal to K we have a corresponding unique orthogonal $M \times K$ matrix $\boldsymbol{B}$.

With the differentiation instruments in hand, we can construct a collection of unconditional moment restrictions. Let $\epsilon_{sj}(\theta, \boldsymbol{B}) = \log W_{sj} + \theta \log p_{sj} + \alpha_j + H_s + \zeta_j^\top \boldsymbol{B}\boldsymbol{B}^\top Z_s$. Then our approach is to define $g_{sj}(\boldsymbol{B}) = \epsilon_{sj}(\theta, \boldsymbol{B}) \odot G(\mathbf{p}, \zeta)$, and require

$$\mathbb{E} g_{sj}(\theta, \boldsymbol{B}) = 0.$$

If $G(\mathbf{p}, \zeta)$ includes a (non-zero) constant as an instrument, then this implies that we also have

$$\mathbb{E}\epsilon_{sj}(\theta, \boldsymbol{B}) = 0.$$

But treating α and H as fixed effects, we can require more; in particular that expected errors are zero for every good j and every market-period s, so that we have

$$\mathbb{E}_s \epsilon_{sj}(\theta, \boldsymbol{B}) = 0 \qquad \text{for all } s \in \mathcal{S}\text{; and} \tag{32}$$

$$\mathbb{E}_j \epsilon_{sj}(\theta, \boldsymbol{B}) = 0 \qquad \text{for all } j = 1, 2, \ldots, J. \tag{33}$$

Either of these (zero for every good; zero for every market-state) implies the weaker condition that the expected value of ϵ be zero, of course. From each of these conditions, we estimate α and H as:

$$\hat{H}_s = \frac{1}{J}\sum_j \log W_{sj} - \theta\frac{1}{J}\sum_j \log p_{sj} - \left(\frac{1}{J}\sum_j \zeta_j^\top\right)\boldsymbol{B}\boldsymbol{B}^\top \sum_{j'} \zeta_{j'} W_{sj'} \tag{34}$$

$$\hat{\alpha}_j = \frac{1}{S}\sum_s \log W_{sj} - \theta\frac{1}{S}\sum_s \log p_{sj} - \zeta_j^\top \boldsymbol{B}\boldsymbol{B}^\top \frac{1}{S}\sum_s\sum_{j'} \zeta_{j'} W_{sj'} - \frac{1}{S}\sum_s \hat{H}_s, \tag{35}$$

finally requiring $\sum_j \hat{\alpha}_j = 0$. Conditional on $(\theta, \boldsymbol{B})$ the parameters $(\alpha, \boldsymbol{B})$ are just identified by these restrictions. Then let

$$\dot{\epsilon}_{sj} = \epsilon_{sj} - H_s - \alpha_j,$$

and define

$$\dot{g}_{sj}(\theta, \boldsymbol{B}) = \dot{\epsilon}_{sj}(\theta, \boldsymbol{B}) \odot G(\mathbf{p}, \zeta)$$

which effectively applies a "within" transformation to eliminate the latent variables H_s and α_j. We then exploit the moment conditions $\mathbb{E}\dot{g}_{sj}(\theta, \boldsymbol{B}) = 0$, or the sample counterparts

$$m_N(\theta, \boldsymbol{B}) = \frac{1}{S}\sum_s \frac{1}{J}\sum_j \dot{g}_{js}(\theta, \boldsymbol{B}).$$

The vector of sample moments m_N has dimension ℓ. We estimate the covariance matrix of these moments using

$$\hat{\Omega}_N(\theta, \boldsymbol{B}) = \frac{1}{S}\sum_s \frac{1}{J}\sum_j \dot{g}_{js}(\theta, \boldsymbol{B})\dot{g}_{js}(\theta, \boldsymbol{B})^\top,$$

and then construct the continuously-updated GMM criterion function

$$J_N(\theta, \boldsymbol{B}) = N m_N(\theta, \boldsymbol{B})^\top \hat{\Omega}_N(\theta, \boldsymbol{B})^{-1} m_N(\theta, \boldsymbol{B}).$$

Our continuously-upated estimator (CUE) is then the solution to

$$\min_{\theta, \boldsymbol{B} \in \mathrm{Orth}(M,K)} J_N(\theta, \boldsymbol{B}), \tag{36}$$

where $\mathrm{Orth}(M, K)$ is the set of orthogonal matrices in $\mathbb{R}^{M\times K}$.

Computation Computing the solution to (36) involves choosing $1 + MK$ parameters which enter the objective function non-linearly, and which are subject to a set of non-linear constraints. If K is small relative to M, this greatly reduces the computational cost, but the non-linear problem is still difficult.

To make the problem manageable, for any fixed value of K we use methods to optimize over the manifold of positive semi-definite matrices described by Boumal (2023), implemented in the the package `pymanopt` (Townsend, Koep, and Weichwald 2016). By searching only over points on this smooth manifold we are able to considerably simplify the optimization problem. Our approach to search exploits trust-region methods applied to manifolds developed by Baker, Absil, and Gallivan (2008), which provides nice guarantees regarding computational complexity and global convergence (Sheng and Yuan 2024).

Given an algorithm for solving (36) for fixed K, we address the global problem of choosing K by solving first for the case of $K = 1$. This yields a value of the criterion J^1, which is asymptotically distributed χ^2 with $\ell - MK - 1$ degrees of freedom under the null hypothesis that $\mathbb{E}m_N(\theta, \hat{\boldsymbol{B}}) = 0$; i.e., that the moment restrictions are satisfied by this model with $K = 1$. If we are able to *reject* this null at standard levels of confidence (1%), we take $K = 2$, and re-estimate, yielding the statistic J^2.

The difference $J^1 - J^2$ is itself distributed χ^2 with M degrees of freedom under the null that $\mathbb{E}J^1 - J^2 = 0$. This leaves us with the following algorithm:

1. Initialize $k = 1$.

2. Fail to reject $K = k$ model. We are finished.

3. Reject $K = k + 1$. Fail to reject $\mathbb{E}J^k - J^{k+1} = 0$. This suggests that we can reject the model, and that an increase in the number of aggregators is *not* significantly improving model fit. We are finished.

4. Reject $K = k + 1$ model. Reject $\mathbb{E}J^k - J^{k+1} = 0$. This suggests that we can reject the model for $K = k + 1$, but that increases in k are producing significant improvements in model fit.

5. Let $k = k + 1$, and go to step 1.

5.4 Preliminary results

Figure 1 provides very preliminary results of estimates of $\overline{\Gamma}$ as defined in equation 29 above, where we also impose a uniform own price effect parameter θ across goods.[14]

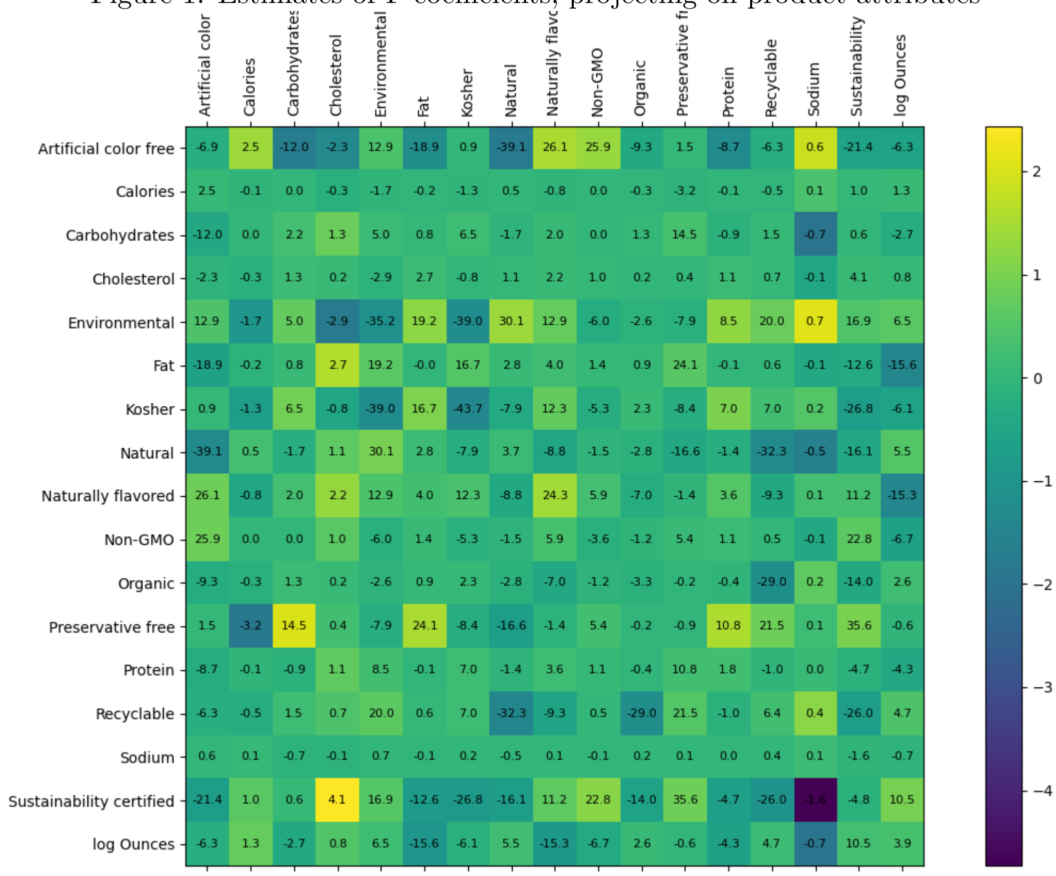
In this specification, we project on 17 product attributes shown in colum and row headings. We impose symmetry of $\overline{\Gamma}$ (linear constraints) but not positive semi-definiteness, and we do not impose symmetry of t-stat estimates (shown in Figure 1 using color codes).

Conclusions

XXX TBD

[14] Our preliminary estimate of θ is: 1.60 with a standard error: 0.008.

Figure 1: Estimates of $\overline{\Gamma}$ coefficients, projecting on product attributes



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Appendix

Lemma from Goldman and Uzawa (1964)

Take a smooth function f and several smooth functions $\{g_k\}_{k\leq K}$, each from $\mathbb{R}^J$ to $\mathbb{R}$. Suppose that there are scalar functions λ_k, $k\leq K$, from $\mathbb{R}^J$ to $\mathbb{R}$ such that:

$$\nabla f(x) \;=\; \sum_k \lambda_k(x)\,.\,\nabla g_k(x)$$

for any x, and suppose that each level set of $\{g_k\}_{k\leq K}$ is connected, then there exists a function G from $\mathbb{R}$ to $\mathbb{R}$ such that:

$$f(x) \;=\; G\big(g_1(x),...,g_K(x)\big)$$

Proof of the lemma It suffices to show that $g_k(x_0)=g_k(x_1)$ (for all k) implies $f(x_0)=f(x_1)$ for any $x_0\in\mathbb{R}^J$ and $x_1\in\mathbb{R}^J$, i.e. that f only takes a single value on a level set of the set of g. Take x_0 and x_1 on such a level set. Since they are on the same level set, following our assumption on connectedness, there is a smooth path $x(t)$ between these two points that remains on the level set, with $x(0)=x_0$ and $x(1)=x_1$. Along that path, define $\phi(t)=f(x(t)$. We have then:

$$\phi'(t) \;=\; \nabla f(\phi(t))\cdot x'(t) \;=\; \lambda(\phi(t))\,.\sum_k \nabla g_k(\phi(t))\cdot x'(t) \;=\; 0$$

which is null since each $\nabla g_k(\phi(t))\cdot x'(t) \;=\; 0$ as $x(t)$ remains on the level set of each g_k. Hence $\phi(1)=\phi(0)$, which means that $f(x_1)=f(x_0)$.

A Proof of Proposition 1

Step 1: functions of K shares and prices. The first step is to show that for each good j, we can write it as a function of its own price x_j and of the first K prices x_k and demand F_k.

Denote by $S|_K$ the S matrix where we include only the first K rows, so that we include only the first K goods $k=1,...,K$. Define $\partial F|_K$ and $\sigma|_K$ similarly. We have then: $\partial F|_K \;=$
$sigma|_K+S|_K$.

The remaining rows of $\partial F-\sigma$ are then linear combinations of the first ones, denoting B_{JK} the matrix that yields the remaining rows depending on S_K. Thus, we can then define matrix

$$B \;=\; \begin{pmatrix} \mathbb{1}_K \\ B_{JK} \end{pmatrix}$$

where $\mathbb{1}_K$ is the identity matrix on $\mathbb{R}^K$, so that matrix B has rank K and we have then:

$$\partial F \;=\; \sigma \;+\; B\;\cdot\;\big(\partial F|_K \;-\;\sigma|_K\big)$$

This means that the gradient of each F_i is a linear combination of the gradients of x_i, the gradients of $x_1,\;...\;,x_K$ as well as the gradients of $F_1,...,F_K$:

$$\frac{\partial F_i}{\partial x_j} \;=\; \sigma_i\mathbb{1}(j=i) \;+\; \sum_k B_{ik}\left[\frac{\partial F_k}{\partial x_j}-\sigma_k\mathbb{1}(j=k)\right]$$

Combined with the connectedness assumption [C4], Lemma 1 of Goldman-Uzawa (1956) [Lemma reproduced above] implies that for each good i there exists a function g_i of $2K+1$ arguments such that we can write:

$$F_i(x) \;=\; g_i\big(x_i,F_1(x),...,F_K(x),x_1,...,x_K\big)$$

Step 2: σ_k and the derivatives of g_i. The rank assumption [C2-ii] on the Jacobian of the arguments of g_i implies that the coefficients in the expression above coincide with the derivatives of function g_i.

First, the derivative in the first argument is $\frac{\partial g_i}{\partial x_i} \;=\; \sigma_i$, which implies that σ_i can be written as a function of $\big(x_i,F_1(x),...,F_K(x),x_1,...,x_K\big)$ for any $i>K$. . Regarding the terms in F_k, we must have:

$$\frac{\partial g_i}{\partial F_k} \;=\; B_{ik}$$

so that B_{ik} can be itself written as a function of $\big(x_i, F_1(x), ..., F_K(x), x_1, ..., x_K\big)$. For the derivatives in x_k, we obtain:

$$\frac{\partial g_i}{\partial x_k} \;=\; -\sigma_k B_{ik}$$

Hence $\frac{\partial g_i}{\partial x_k}$ can also be written as functions of $\big(x_i, F_1(x), ..., F_K(x), x_1, ..., x_K\big)$. Taking ratios, σ_k can in turn be written as functions of $\big(x_i, F_1(x), ..., F_K(x), x_1, ..., x_K\big)$. This can be done for all i such that $B_{ik} \neq 0$ is non-null.

Under assumption [C2-i], we must have at least two goods $i > K$ and $j > K$ such that $B_{ik} \neq 0$ and $B_{jk} \neq 0$, i.e. $\frac{\partial g_i}{\partial F_k} \neq 0$ and $\frac{\partial g_j}{\partial F_k} \neq 0$. If that wasn't the case, e.g. if all the B_{ik} cells are zero for a specific k, then one of the columns of the B_{JK} matrix would be null, and its rank would be strictly smaller than K after we remove the $k \leq K$ rows. If the B_{ik} cells are all zero for a specific k, aside from a single i good, then the same reasoning would apply if we remove the row corresponding to that good i. Hence there exist at least two non-zero entries B_{ik} and B_{jk} for any specific column k.

We then obtain that σ_k can be written both as function of $\big(x_i, F_1(x), ..., F_K(x), x_1, ..., x_K\big)$ and as a function of $\big(x_j, F_1(x), ..., F_K(x), x_1, ..., x_K\big)$. Under assumption [C2-ii], linear independent of their Jacobians implies that σ_k only depends on the common set of arguments, i.e. $\big(F_1(x), ..., F_K(x), x_1, ..., x_K\big)$. We use this result in the next step.

Step 3: defining flows Ψ_k. For any good $k \leq K$, define the "flow" $\Psi_k\big(t\big|F_1, ..., F_K, x_1, ..., x_K\big)$ as a function from $\mathbb{R} \times \mathbb{R}^{2K}$ to $\mathbb{R}^{2K}$ such that:

$$\Psi_k\big(0\big|F_1, ..., F_K, x_1, ..., x_K\big) \;=\; \big(F_1, ..., F_K, x_1, ..., x_K\big)$$

and:

$$\frac{\partial \Psi_{k,xk'}}{\partial t} \;=\; 0 \qquad ; \qquad \frac{\partial \Psi_{k,Fk'}}{\partial t} \;=\; 0$$

whenever $k' \neq k$, i.e. t does not lead to changes in values w.r.t coordinates $F_{k'}$ and $x_{k'}$ for $k' \neq k$), while:

$$\frac{\partial \Psi_{k,xk}}{\partial t} \;=\; 1 \qquad ; \qquad \frac{\partial \Psi_{k,Fk}}{\partial t} \;=\; \sigma_k\big(\Psi(t)\big) \;<\; 0$$

which means that as time t increases we also increase the price x_k of good k and its corresponding demand F_k by a corresponding amount $\sigma_k < 0$. In this transformation, since the sign of σ_k does not change, there is a unique coordinate in F_k for each coordinate x_k.

This is an ordinary differential equation that admits a solution. Moreover, it is defined for all $t \in \mathbb{R}$. If the flow Ψ_k was defined only up to an upperbound T, if would leave any compact as t approaches T. Since x would remain bounded (x_i remains constant for any $i \neq k$ and x_k would increase by at most T from $t = 0$), and F_i remains constant for any $i \neq k$, it must be that $|F_k|$ goes to infinity for some k. Under assumption [C3], this leads to a contradition as other F_i could then nor remain constant.

Alternatively, note that the flows would be also globally defined under the alternative assumption that σ_k be bounded.

Step 4: invariance property. We can then check that each demand function g_j are invariant to transformations along flows Ψ_k. Take any good j and define:

$$F_j(t) \;=\; g_j\big(x_{j0}, \Psi_k\big(t\big|F_1, ..., F_K, x_1, ..., x_K\big)\big)$$

where we hold the first argument fixed to some arbitrary value x_{j0}. We obtain, using the derivatives of flow Ψ_k:

$$\frac{\partial F_j}{\partial t} \;=\; \sigma_k(\Psi_k)\frac{\partial g_i}{\partial F_k}(\Psi_k) \;+\; \frac{\partial g_i}{\partial x_k}(\Psi_k) \;=\; 0$$

which is equal to zero given the earlier finding on the derivatives $\frac{\partial g_i}{\partial x_k}$ and $\frac{\partial g_i}{\partial F_k}$.

Step 5: commutative frame. We can then show that the set of flows $\{\Psi_k\}_{k\leq K}$ commute, i.e. it is equivalent to shift first by t_k using the flow Φ_k and then shift by $t_{k'}$ using flow $\Phi_{k'}$ or vice versa:

$$\Psi_{k'}\big(+t_{k'}\big|\Psi_k\big(+t_k\big|\{F,x\}_K\big)\big) = \Psi_k\big(+t_k\big|\Psi_{k'}\big(+t_{k'}\big|\{F,x\}_K\big)\big)$$

where $\{F,x\}_K \equiv (F_1, ..., F_K, x_1, ..., x_K) \in \mathbb{R}^{2K}$.

The reason for this is that these flows remain on the same level sets functions g_i's, holding constant their first argument (thus, they are "integrable"). Take some fixed values $\bar{x}_i$ for $i > K$ and define function G from $\mathbb{R}^{2K}$ to $\mathbb{R}^{J-K}$ as:

$$G\big(\{F,x\}_K\big) \;=\; \big\{g_i(\bar{x}_i, \{F,x\}_K\big\}_{i>K}$$

Three remarks about the Jacobian are then relevant: i) first, the Jacobian of G in F (holding x constant) coincides with matrix B_{JK} defined earlier, which has a rank K; ii) each column of the Jacobian of G in x (keeping F constant) is colinear with the k'th column of B, hence again the Jacobian of G in x; iii) given the colinearity, the rank of G in (F, x) combined is exactly K again.

Hence we can apply the "constant-rank level set theorem" (see e.g. Lee 2003, chapter 5, theorem 5.1) telling us that each level set of G in $\{F, x\}$ is a properly embedded submanifold of dimension K in $\mathbb{R}^{2K}$. Moreover, since the differential in F has rank K, locally there is a unique F on the level set, given x. Similarly, there is a unique x on the level set, conditional on F. Hence, locally, there is a one-to-one mapping between x and F such that $\{F, x\}$ remains on the level set.

Then, notice that the x coordinates of $\Psi_{k'}\big(+t_{k'}\big|\Psi_k\big(+t_k\big|\{F,x\}_K\big)\big)$ are the same as those of $\Psi_k\big(+t_k\big|\Psi_{k'}\big(+t_{k'}\big|\{F,x\}_K\big)\big)$ since we are both shifting x_k by $+t_k$ and $x_{k'}$ by $+t_{k'}$ while keeping other x's constant. Since both remain on the same level set of G, the remaining coordinates in F must be the same in both cases, showing that the two flows commute.[15]

We can then simply define $\Psi\big(t_1, ..., t_K\big|\{F,x\}_K\big)$ as the combination of shifts along each flow Ψ_k by t_k without referring to the ordering. Using this "commutative frame", it is then easy to define aggregators and demand as a function of these aggregators.

Step 6: aggregators and demand function. We define aggregators Λ as the F component of Ψ, where we shift the initial $\{F, x\}_K$ by $t = -x$:

$$\big(\Lambda_1(x), ..., \Lambda_K(x), 0, ...0\big) \;=\; \Psi\big(-x_1, ..., -x_K\big|\big(F_1(x), ..., F_K(x), x_1, ..., x_K\big)\big)$$

or equivalently defining each Λ_k as: $\Lambda_k(x) \;=\; \Psi_{k,Fk}\big(-x_k\big|\big(F_1(x), ..., F_K(x), x_1, ..., x_K\big)\big)$. By pushing back by $+x_k$ (inverting the flow), notice that we have:

$$\big(F_1(x), ..., F_K(x), x_1, ..., x_K\big) \;=\; \Psi\big(x_1, ..., x_K\big|\big(\Lambda_1(x), ..., \Lambda_K(x), 0, ...0\big)\big)$$

so we can express each F_k as a function of x_k and the Λ's:

$$F_k(x) \;=\; \Psi_{Fk}\big(x_k\big|\big(\Lambda_1(x), ..., \Lambda_K(x), 0, ...0\big)\big)$$

This gives Theorem 1 for goods $k \leq K$. where the D_k function is given by Ψ_{Fk}.

For goods $i > K$, using the invariance property (step 4), we can see that we can replace each demand F_k as a argument by $\Lambda_k(x)$ and each price x_k by 0:

$$\begin{aligned}
F_i(x) &= g_i\big(x_i, F_1(x), ..., F_K(x), x_1, ..., x_K\big) \\
&= g_i\big(x_i, \Psi\big(-x_1, ..., -x_K\big|\big(F_1(x), ..., F_K(x), x_1, ..., x_K\big)\big)\big) \\
&= g_i\big(x_i, \Lambda_1(x), ..., \Lambda_K(x), 0, ..., 0\big)
\end{aligned}$$

Denoting D_i the function g_i by dropping the $0, ...0$ argument, we obtain the result from Theorem 1, i.e. that we can express demand as a function of its own price x_i and K aggregators:

$$F_i = D_i\big(x_i, \Lambda_1(x), ..., \Lambda_K(x)\big) \equiv g_i\big(x_i, \Lambda_1(x), ..., \Lambda_K(x), 0, ..., 0\big)$$

[15] Note that this argument is local, but flows that commute locally also commute globally.

B Proof of Proposition 3

Here, to lighten the notation, we denote $x_i = \log p_i$.

Step 1: Separability

We start from the integrability assumption:

$$\frac{\partial P}{\partial x_i} = W_i(x_i, \Lambda(x)) \tag{37}$$

Denote $S_i(x_i, \Lambda) = \int_0^{x_i} W_i(t, \Lambda)dt$ a primitive of W_i in x_i, and denote:

$$S(x) = \sum_j S_j(x_i, \Lambda(x))$$

the sum of S_i's, evaluated at $\Lambda = \Lambda(x)$. Define

We have then:

$$\frac{\partial S}{\partial x_i} = \frac{\partial P}{\partial x_i} + \sum_k \left(\sum_j \frac{\partial S_j}{\partial \Lambda_k} \right) \frac{\partial \Lambda_k}{\partial x_i} \tag{38}$$

Hence the gradient of $S - P$ is colinear with the gradients of the aggregators Λ. Using Lemma 1, we can thus express $S - P$ as a function of the aggregators, using also the assumption that iso-Λ surfaces are connected. Hence:

$$S(x) - P(x) = M(\Lambda(x)) \tag{39}$$

for some function M, and thus:

$$\sum_j S_j(x_i, \Lambda(x)) \; = \; M(\Lambda(x)) \; + \; P(x) \tag{40}$$

Step 2: Implication of the rank of $\frac{\partial \Lambda_k}{\partial x_i}$

Differentiating the last equality above, and using $\frac{\partial P}{\partial x_i} = \frac{\partial S_i}{\partial x_i}$, we have:

$$\sum_k \left(\sum_j \frac{\partial S_j}{\partial \Lambda_k} \right) \frac{\partial \Lambda_k}{\partial x_i} \; = \; \sum_k \left(\frac{\partial M}{\partial \Lambda_k} \right) \frac{\partial \Lambda_k}{\partial x_i}$$

Given the assumption that the collection of vectors $\frac{\partial \Lambda_k}{\partial x_i}$ has rank K, it must be that:

$$\sum_j \frac{\partial S_j}{\partial \Lambda_k} \; = \; \frac{\partial M}{\partial \Lambda_k} \tag{41}$$

for each Λ_k.

Step 3: Symmetry and invertibility

Differentiating the last equality above (FOC in Λ_k) w.r.t x_i, we obtain:

$$\sum_j \frac{\partial^2 S_j}{\partial x_i \partial \Lambda_k} \; + \; \sum_{k'=0} \sum_j \frac{\partial^2 S_j}{\partial \Lambda_{k'} \partial \Lambda_k} \frac{\partial \Lambda_{k'}}{\partial x_i} \; = \; \sum_{k'=0} \frac{\partial^2 M}{\partial \Lambda_{k'} \partial \Lambda_k} \frac{\partial \Lambda_{k'}}{\partial x_i}$$

But notice that $\sum_j \frac{\partial^2 S_j}{\partial x_i \partial \Lambda_k} \; = \; \frac{\partial W_i}{\partial \Lambda_k}$ so

$$\frac{\partial W_i}{\partial \Lambda_k} \; + \; \sum_{k'=1} \sum_j \frac{\partial^2 S_j}{\partial \Lambda_{k'} \partial \Lambda_k} \frac{\partial \Lambda_{k'}}{\partial x_i} \; = \; \sum_{k'=1} \frac{\partial^2 M}{\partial \Lambda_{k'} \partial \Lambda_k} \frac{\partial \Lambda_{k'}}{\partial x_i}$$

This equality can be rewritten as:

$$\frac{\partial W_i}{\partial \Lambda_k} \; = \; \sum_{k'=1} \Gamma_{kk'} \frac{\partial \Lambda_{k'}}{\partial x_i} \tag{42}$$

where Γ is a symmetric $K \times K$ matrix (given the symmetry of the cross derivatives) with coefficients:

$$\Gamma_{kk'} = \frac{\partial^2 M}{\partial \Lambda_{k'} \partial \Lambda_k} - \sum_j \frac{\partial^2 S_j}{\partial \Lambda_{k'} \partial \Lambda_k} \tag{43}$$

Invertibility Since we also assume that $\left\{ \frac{\partial W_i}{\partial \Lambda_k} \right\}$ has full rank K, we obtain that Γ matrix must be invertible, with its inverse denoted by γ (also symmetric). It implies that the gradients of Λ must be a linear combination of the partial derivatives of W_i:

$$\frac{\partial \Lambda_k}{\partial x_i} = \sum_{k'=1} \gamma_{kk'} \frac{\partial W_i}{\partial \Lambda_{k'}} \tag{44}$$

Step 4: Using the budget constraint

The budget constraint condition imposes $\sum_i W_i(x_i, \Lambda(x)) = 1$ when the aggregators Λ are evaluated at x. Differentiating , we find:

$$-\frac{\partial W_j}{\partial x_j} = \sum_{k=1} \left(\sum_i \frac{\partial W_i}{\partial \Lambda_k} \right) \frac{\partial \Lambda_k}{\partial x_j} \tag{45}$$

Hence $\frac{\partial W_j}{\partial x_j}$ is colinear with the set of vectors $\frac{\partial \Lambda_k}{\partial x_j}$. Denote by $\gamma_{kk'}$ the coefficients of the inverse of Γ. We obtain:

$$-\frac{\partial W_j}{\partial x_j} = \sum_{k'} \sum_k \left(\sum_i \frac{\partial W_i}{\partial \Lambda_k} \right) \gamma_{kk'} \frac{\partial W_j}{\partial \Lambda_{k'}}$$

Dividing by W_j, and denoting

$$v_k = \sum_{k'} \gamma_{kk'} \left(\sum_i \frac{\partial W_i}{\partial \Lambda_{k'}} \right), \tag{46}$$

we have then:

$$-\frac{\partial \log W_j}{\partial x_j} = \sum_k v_k \frac{\partial \log W_j}{\partial \Lambda_k} \tag{47}$$

where the last derivatives are taken by holding V and x_i constant, respectively. This colinearity between derivatives of W_j is crucial to obtain the functional form for W_j. Before getting into integrating this differential equation, some additional work on v_k is needed.

Step 5: Coefficients v_k as functions of Λ

Take the derivative of the above equation with respect to p_i for a good $i \neq j$. Since $\frac{\partial \log W_j}{\partial x_j}$ and $\frac{\partial \log W_j}{\partial \Lambda_k}$ only depends on the aggregators and x_j, we obtain:

$$-\sum_{k'} \frac{\partial^2 \log W_j}{\partial x_j \partial \Lambda_{k'}} \frac{\partial \Lambda_{k'}}{\partial x_i} = \sum_k \frac{\partial v_k}{\partial x_i} \frac{\partial \log W_j}{\partial \Lambda_k} + \sum_{k,k'} v_k \frac{\partial^2 \log W_i}{\partial \Lambda_k \partial \Lambda_{k'}} \frac{\partial \Lambda_{k'}}{\partial x_i} \tag{48}$$

Rearranging, and multiplying by W_j, we get:

$$-\sum_{k'} \frac{\partial v_{k'}}{\partial x_i} \frac{\partial W_j}{\partial \Lambda_{k'}} = \sum_{k'} \frac{\partial \Lambda_{k'}}{\partial x_i} B_{jk'} \tag{49}$$

where for some $B_{jk'} = \sum_k v_k \frac{\partial^2 W_j}{\partial \Lambda_k \partial \Lambda_{k'}} + \frac{\partial^2 W_j}{\partial x_j \partial \Lambda_{k'}}$ (this notation B will not appear again).

As we assume that $\frac{\partial W_i}{\partial \Lambda_k}$ have full rank K even if we drop a good j, we obtain that the gradients $\frac{\partial v_k}{\partial x_i}$ are colinear with the collection of gradients $\left\{ \frac{\partial \Lambda_{k'}}{\partial x_i} \right\}$. Lemma 1 implies that each v_k can be written as a function of Λ.

$$v_k = v_k(\Lambda) \tag{50}$$

Step 6: Flow Φ

Functional form equations for W_i. Recall that: $\frac{\partial \log W_j}{\partial x_j} = -\sum_{k=1} v_k(\Lambda) \frac{\partial \log W_j}{\partial \Lambda_k}$ where $v_k(\Lambda)$ is a function of aggregators Λ. We obtain:

$$-\frac{\partial W_i}{\partial x_i} = \sum_k v_k(\Lambda) \frac{\partial W_i}{\partial \Lambda_k} \tag{51}$$

for each good i.

Defining the flow Φ. A solution to these equations is based on the existence of a mapping for each t from Λ into a new vector of aggregators $\Phi(t, \Lambda)$ such that $\Phi(0, \Lambda) = \Lambda$ and such that:

$$\frac{\partial \Phi_k}{\partial t} = v_k(\Phi) \tag{52}$$

Action of Φ on W_i. This flow can be used to highlight symmetries across goods and invariances in the demand functions W_i and function M defined in step 1. First, consider $W_i(x_i + t, \Phi(t, \Lambda))$ as a function of t. Its derivative in t is given by:

$$\begin{aligned} & \frac{\partial W_i}{\partial x_i}(x_i + t, \Phi(t,\Lambda)) + \sum_{k=1} \frac{\partial \Phi_k}{\partial t}(\Phi(t,\Lambda)) \frac{\partial W_i}{\partial \Lambda_k}(x_i + t, \Phi(t,\Lambda)) \\ = \; & \frac{\partial W_i}{\partial x_i}(x_i + t, \Phi(t,\Lambda)) + \sum_{k=1} v_k(\Phi(t,\Lambda)) \frac{\partial W_i}{\partial \Lambda_k}(x_i + t, \Phi(t,\Lambda)) \;\; = \;\; 0 \end{aligned}$$

hence it does not depend on t, which implies:

$$W_i(x_i + t, \Phi(t, \Lambda)) = W_i(x_i, \Lambda) \tag{53}$$

for any t, x_i and Λ. Another way to highlight the role of Φ is to see that it captures the price effects and reduces demand to a function of aggregators after adjusting that price effect:

$$W_i(x_i, \Lambda) = W_i(0, \Phi(-x_i, \Lambda)) \tag{54}$$

Note also that $W_i(x_i, \Phi(t, \Lambda)) = W_i(x_i - t, \Lambda)$ strictly increases with t since W_i decreases in x_i.

Maximal flow. We still need to check that Φ is a global flow, i.e. defined for all $t \in \mathbb{R}$ (i.e. the vector field v is "complete").

By contradiction, suppose that for some Λ_0 the flow $\Phi(t, \Lambda_0)$ is defined only up to T. We would have:

$$\lim_{t \to T} W_i(0, \Phi(t, \Lambda_0)) = \lim_{t \to T} W_i(-t, \Lambda_0) = W_i(-T, \Lambda_0) > 0$$

for all i.

However, if the flow is defined only up to T, $\Phi([0,T), \Lambda_0)$ of $[0,T)$ cannot be contained into a compact set ("Escape Lemma", see Lemma 9.19 in Lee 2013's Intro to Smooth Manifolds). Given our assumption ("no escaping"), this implies that $\max_i |\log W_i(0, \Phi(t, \Lambda_0))|$ is unbounded and cannot has a finite limit, and contradicts the results above.

Other properties. Notice that $\Phi(t, \Phi(t', \Lambda)) = \Phi(t + t', \Lambda)$ for any t and t', hence, for any given t, Φ it is invertible in Λ, with inverse $\Phi(-t, \Lambda)$, given that $\Phi(t, \Phi(-t, \Lambda)) = \Lambda$. Also, for any t, Φ is differentiable (hence a diffeomorphism).

Step 7: Projecting aggregators

Here the goal is to show that each pair Λ can be written as:

$$\Lambda \; = \; F(\Lambda^*, \Lambda')$$

for some isomorphism $F : \mathbb{R} \times \mathcal{M}_0 \to \mathbb{R}^K$ for some submanifold $\mathcal{M}_0$, such that we have a canonical projection on flow Φ, i.e. such that:

$$\Phi(t, F(\Lambda^*, \Lambda')) \; = \; F(\Lambda^* + t, \Lambda') \tag{55}$$

for any t, Λ^* and Λ'.

Total price effects. Define the function D_0 of Λ by evaluating the sum of D_i at a reference point $x_i = 0$ for each good:

$$W_0(\Lambda) = \sum_i W_i(0, \Lambda)$$

Since for each i we obtain that $W_i(0, \Phi(t, \Lambda)) = W_i(-t, \Lambda)$ strictly increases with t, we also obtain that $W_0(\Phi(t, \Lambda))$ strictly increases with t.

We define by $\mathcal{M}_0$ the set of Λ such that $W_0(\Lambda) = 1$:

$$\mathcal{M}_0 \;=\; W_0^{-1}(\{1\}) \tag{56}$$

As $W_0(\Phi(t, \Lambda))$ strictly increases with t, we can deduce that the vector field v is never tangent to $\mathcal{M}_0$. This will be useful to apply the Flowout Theorem (see below).

Defining the mapping. We then simply construct F such that

$$F(\Lambda^*, \Lambda') \;=\; \Phi(-\Lambda^*, \Lambda')$$

As the flow Φ is global (and differentiable), F is defined for all Λ^* and Λ'. We have yet to show surjectivity and injectivity globally.

Surjectivity. Take $\Lambda \in \mathbb{R}^K$. We need to find $\Lambda' \in \mathcal{M}_0$ and $\Lambda^* \in \mathbb{R}$ such that $F(\Lambda^*, \Lambda') = \Lambda$. But notice that we have then:

$$\Lambda' = \Phi(\Lambda^*, \Lambda)$$

So, for such a Λ^* and Λ' to exist, we need to show that

$$W_0(\Phi(\Lambda^*, \Lambda)) \;=\; 1$$

for some Λ^*. Note that:

$$W_0(\Phi(\Lambda^*, \Lambda)) \;=\; \sum_i W_i(0, \Phi(\Lambda^*, \Lambda)) \;=\; \sum_i W_i(-\Lambda^*, \Lambda)$$

One of our assumption (on "Total price effects") is that for any Λ and any $y > 0$, there exist a real $t \in \mathbb{R}$ such that:

$$\sum_i W_i(t, \Lambda) = 1 \tag{57}$$

We can thus find Λ^* such that:

$$\sum_i W_i(-\Lambda^*, \Lambda) = 1$$

which is equivalent to having $\Phi(\Lambda^*, \Lambda)) \in \mathcal{M}_0$. Setting $\Lambda' = \Phi(\Lambda^*, \Lambda))$, we have $\Lambda = \Phi(-\Lambda^*, \Lambda') = F(\Lambda^*, \Lambda')$.

Injectivity. We also need to show that F is globally injective, but this is relatively easy using function D_0. Consider two sets of aggregators, $(\Lambda_1^*, \Lambda_1')$ vs. $(\Lambda_0^*, \Lambda_0')$, we have then:

$$\begin{aligned} F(\Lambda_1^*, \Lambda_1') \;=\; F(\Lambda_0^*, \Lambda_0') \quad &\Longleftrightarrow \quad \Phi(-\Lambda_1^*, \Lambda_1') = \Phi(-\Lambda_0^*, \Lambda_0') \\ &\Longleftrightarrow \quad \Phi(\Lambda_0^* - \Lambda_1^*, \Lambda_1') = \Lambda_0' \end{aligned}$$

This implies that $D_0(\Phi(\Lambda_0^* - \Lambda_1^*, \Lambda_1')) = 1$. Since D_0 is strictly monotonic in Λ^*, and since $D_0(\Phi(0, \Lambda_1')) = 1$, we obtain that Λ_1^* must be equal to Λ_0^*. In turn, we get: $\Lambda_0' = \Phi(\Lambda_0^* - \Lambda_1^*, \Lambda_1') = \Phi(0, \Lambda_1') = \Lambda_1'$.

Implication for D_i. These results imply that we can write:

$$W_i(x_i, F(\Lambda^*, \Lambda')) \;=\; W_i\big(x_i, \Phi(-\Lambda^*, \Lambda')\big) \;=\; W_i\big(0, \Phi(-x_i - \Lambda^*, \Lambda')\big) \;=\; W_i\big(x_i + \Lambda^*, \Lambda'\big) \tag{58}$$

Hence, up to a isomorphic mapping of the aggregators, we can rewrite W_i as a function of the price shifter Λ^* and a vector of aggregators that belongs to a submanifold of lower dimension.

Step 8: Implications for M and V. As described earlier, evaluating aggregators at x, we must have:

$$\sum_j S_j(x_i, \Lambda(x)) \; = \; M(\Lambda(x)) \; + \; P(x) \tag{59}$$

$$P(x) \; = \; \sum_j S_j(x_i, \Lambda(x)) \; - \; M(\Lambda(x)) \tag{60}$$

and the first order condition (41) in each aggregator Λ_k implies that we must have:

$$\sum_j \frac{\partial S_j}{\partial \Lambda_k} \; = \; \frac{\partial M}{\partial \Lambda_k} \tag{61}$$

The same sets of FOC can be applied to (Λ^*, Λ') if we use the canonical representation of aggregators.

Note that $S_i(x_i, \Lambda) = \int_0^{x_i} W_i(t, \Lambda) dt$. Hence, using the new functional form based on D_i and Λ^*, we obtain:

$$S_i(x_i, \Lambda^*, \Lambda') = \int_0^{x_i + \Lambda^*} W_i(t, \Lambda') dt$$

The first order condition in Λ^* implies that we must have:

$$\frac{\partial M}{\partial \Lambda^*} \; = \; \sum_j \frac{\partial S_j}{\partial \Lambda^*} \; = \; \sum_j W_j(x_i + \Lambda^*, \Lambda') \; = \; 1$$

Hence:

$$M(\Lambda^*, \Lambda') \; = \; G(\Lambda') + \Lambda^*$$

for some function G that is independent of Λ^*.

We obtain:

$$P(x) \; = -M(\Lambda^*(x), \Lambda'(x)) + S(x) \; = \; -G(\Lambda') - \Lambda^* + \sum_i \int_0^{x_i + \Lambda^*} W_i(t, \Lambda') dt$$

where the aggregators $\Lambda^* = \Lambda^*(x)$ and $\Lambda' = \Lambda'(x)$ are such that the derivatives of the RHS in Λ^* and Λ' are null.

Homogeneity for aggregators. Based on the FOC for aggregators, we have then:

$$\Lambda^*(x + a) \; = \; \Lambda^*(x) + a \quad \text{and} \quad \Lambda'(x + a) \; = \; \Lambda'(x)$$

C Proof of Lemma 4

Suppose that the function $\widetilde{P}(p, \Lambda^*, \Lambda)$ is defined as in equation (6):

$$\log \widetilde{P}(p, \Lambda^*, \Lambda) \;=\; \log \Lambda^* \;-\; G(\Lambda) \;+\; \sum_j \int_{t=1}^{p_j/\Lambda^*} D_j(t, \Lambda)\, d\log t$$

Notice that we have:

$$\left.\frac{\partial \log \widetilde{P}}{\partial \log p_i}\right|_{\Lambda^*, \Lambda} \;=\; D_j(p_j/\Lambda^*, \Lambda)$$

which is positive and decreasing in p_i, hence $\log \widetilde{P}$ is strictly concave in $\log p$, conditional on Λ and Λ^*.

As described, we consider two cases:

i) If $\log \widetilde{P}(p, \Lambda^*, \Lambda)$ is convex in Λ, define $\log \widetilde{\widetilde{P}}(p) = \{\min_\Lambda \log \widetilde{P}(p, 1, \Lambda)\}$.

ii) If $\log \widetilde{P}(p, \Lambda^*, \Lambda)$ is concave in $(\Lambda, \log p)$, define $\log \widetilde{\widetilde{P}}(p) = \max_\Lambda \log \widetilde{P}(p, 1, \Lambda)$.

In each of these two cases, the max or min operations preserve the concavity property, so we obtain that $\log \widetilde{\widetilde{P}}(p)$ is strictly concave in $\log p$. We then define:

$$\log P(p) \;=\; \max_{\Lambda^*}\{\log \Lambda^* + \log \widetilde{\widetilde{P}}(p/\Lambda^*)\}$$

which coincides with the function $P(p)$ defined in the statement of Lemma 4.

Using the fact that $\log \widetilde{\widetilde{P}}$ is strictly concave in $\log p$, we can show that $\log P$ is concave in p. We do this by examining the Hessian, and showing that it is semi-definite negative. Since the right-hand side is strictly concave in $\log \Lambda^*$, aggregator Λ^* is uniquely defined. The first-order condition is:

$$1 \;=\; \sum_i \frac{\partial \log \widetilde{\widetilde{P}}(p/\Lambda^*)}{\partial \log p_i}$$

Differentiating, we get:

$$0 \;=\; -\left(\sum_{i,l} \frac{\partial^2 \log \widetilde{\widetilde{P}}}{\partial \log p_i \partial \log p_l}\right) \frac{\partial \Lambda^*}{\partial \log p_j} \;+\; \sum_i \frac{\partial^2 \log \widetilde{\widetilde{P}}}{\partial \log p_i \partial \log p_j}$$

In matrix form, denoting by $\mathcal{H}$ the Hessian of $\log \widetilde{\widetilde{P}}$, this yields:

$$0 \;=\; -(\mathbb{1}^t \mathcal{H} \mathbb{1}) \nabla \Lambda^{*t} \;+\; \mathbb{1}^t \mathcal{H}$$

and thus: $\nabla \Lambda^{*t} = \eta \mathbb{1}^t \mathcal{H}$ where $\eta < 0$ denotes $1/(\mathbb{1}^t \mathcal{H} \mathbb{1})$.

Turning to P and using again the envelope theorem, we obtain: $\frac{\partial P}{\partial p_i} = \frac{P(p/\Lambda^*)}{p_i} \frac{\partial \log \widetilde{\widetilde{P}}(p/\Lambda^*)}{\partial \log p_i}$ and thus the Hessian:

$$\frac{\partial^2 \log \widetilde{\widetilde{P}}}{\partial p_i \partial p_j} \;=\; \frac{P}{p_i p_j}\left[\frac{\partial \log \widetilde{\widetilde{P}}}{\partial \log p_i} \frac{\partial \log \widetilde{\widetilde{P}}}{\partial \log p_j} \;-\; \mathbb{1}(i=j) \frac{\partial \log \widetilde{\widetilde{P}}}{\partial \log p_i}\right] \;+\; \frac{P}{p_i p_j}\left[\frac{\partial^2 \log \widetilde{\widetilde{P}}}{\partial \log p_i \partial \log p_j} \;-\; \sum_l \frac{\partial^2 \log \widetilde{\widetilde{P}}}{\partial \log p_i \partial \log p_l} \frac{\partial H}{\partial \log p_j}\right]$$

In this expression, the term in first brackets are the coefficients of a semi-definite negative matrix because the diagonal coefficients are negative and weakly "dominate" the non-diagonal coefficients, since its row sum (or column sum) is equal to zero.

Next we show that the terms in the second brackets are also the coefficients of semi-definite negative matrix. In matrix form, the terms in the second matrix coincide with the matrix M defined as:

$$M \;=\; \mathcal{H} - \mathcal{H}\mathbb{1}\nabla \Lambda^{*t} \;=\; \mathcal{H} - \eta \mathcal{H}\mathbb{1}\mathbb{1}^t \mathcal{H}$$

To prove negative semi-definiteness, we show that for any vector v, we have $v^t M v \leq 0$. Indeed, we have:

$$v^t M v \leq 0 \qquad \Leftrightarrow \qquad (v^t \mathcal{H} v) \leq \eta (v^t \mathcal{H} \mathbb{1})(\mathbb{1}^t \mathcal{H} v) \qquad \Leftrightarrow \qquad (\mathbb{1}^t \mathcal{H} \mathbb{1})(v^t \mathcal{H} v) \geq (v^t \mathcal{H} \mathbb{1})^2$$

The latter is Cauchy-Schwartz inequality, which holds for any semi-definite matrix $\mathcal{H}$

D Proof of Proposition 5

While Lemma 4 proves that the demand system is rational and well-defined (under assumptions of Lemma 4). This implies that there exists a well-defined concave utility function $U(q)$ that generates such demand system. By definition, under homothetic preferences, the price index can be obtained from utility as

$$\log P(p) \quad = \quad -\max_{q_i} \Big\{ \log U(q) \quad s.t. \quad \sum_i p_i q_i = 1 \Big\}$$

But we can then show that:

$$\max_{q_i} \Big\{ \log U(q) \quad s.t. \quad \sum_i p_i q_i = 1 \Big\} \quad = \quad \max_q \Big\{ 1 \, - \, \sum_i p_i q_i \, + \, \log U(q) \Big\}$$

To see this equality, notice that in the right, the optimum in q satisfies: $p_i = \frac{\partial \log U(q)}{\partial q_i}$. On the left, we have $\lambda p_i = \frac{\partial \log U(q)}{\partial q_i}$ where λ is the Lagrange multiplier. But since $U(q)$ is homogeneous of degree one, we must have:

$$1 \; = \; \sum_i p_i q_i \; = \; \sum_i q_i \frac{\partial \log U}{\partial q_i} / \lambda \; = \; 1/\lambda$$

So $\lambda = 1$ and the optimal q are the same in both maximization problems. This implies that, on the right, $\sum_i p_i q_i = 1$ at the optimum.

Combining, ignoring a constant term (-1), this implies that the log price index is the conjugate of log utility:

$$\log P(p) \; = \; \min_q \Big\{ \sum_i p_i q_i \; - \; \log U(q) \Big\}$$

Hence, applying the Legendre-Fenchel duality theorem, we can obtain log utility as the concave congugate of the log price index. Denote $S_i(p_i, \Lambda) = \int_{t=0}^{\log p_i} D_i(t, \Lambda)\, dt$ and $u_i(q_i, \Lambda) = \min_p \{p_i q_i - S_i(p_i, \Lambda)\}$ its conjugate in p_i (i.e. conditional on Λ). In the case where $\log \widetilde{P}(p, \Lambda^*, \Lambda)$ is convex in Λ (condition i) of Lemma 4), we get:

$$\begin{aligned}
\log U(q) \quad &= \quad \min_p \Big\{ \sum_i p_i q_i \; - \; \log P(p) \Big\} \\
&= \quad \min_p \Big\{ \sum_i p_i q_i \; - \; \max_{\Lambda^*} \max_{\Lambda} \Big\{ \sum_i S_i(p_i/\Lambda^*, \Lambda) \; + \; \log \Lambda^* \; - \; G(\Lambda) \Big\} \Big\} \\
&= \quad \min_p \min_{\Lambda^*} \min_{\Lambda} \Big\{ \sum_i p_i q_i \; - \; \sum_i S_i(p_i/\Lambda^*, \Lambda) \; - \; \log \Lambda^* \; + \; G(\Lambda) \Big\} \\
&= \quad \min_{\Lambda^*} \min_{\Lambda} \min_p \Big\{ \sum_i p_i q_i \; - \; \sum_i S_i(p_i/\Lambda^*, \Lambda) \; - \; \log \Lambda^* \; + \; G(\Lambda) \Big\} \\
&= \quad \min_{\Lambda^*} \min_{\Lambda} \Big\{ \sum_i \min_p \Big\{ p_i q_i - S_i(p_i/\Lambda^*, \Lambda) \Big\} \; - \; \log \Lambda^* \; + \; G(\Lambda) \Big\} \\
&= \quad \min_{\Lambda^*} \min_{\Lambda} \Big\{ \sum_i u_i(q_i \Lambda^*, \Lambda) \; - \; \log \Lambda^* \; + \; G(\Lambda) \Big\}
\end{aligned}$$

In the concave case (condition ii) of Lemma 4), we have:

$$\begin{aligned}
\log U(q) \quad &= \quad \min_p \Big\{ \sum_i p_i q_i \; - \; \max_{\Lambda^*} \min_{\Lambda} \Big\{ \sum_i S_i(p_i/\Lambda^*, \Lambda) \; + \; \log \Lambda^* \; - \; G(\Lambda) \Big\} \Big\} \\
&= \quad \min_p \min_{\Lambda^*} \max_{\Lambda} \Big\{ \sum_i p_i q_i \; - \; \sum_i S_i(p_i/\Lambda^*, \Lambda) \; - \; \log \Lambda^* \; + \; G(\Lambda) \Big\} \\
&= \quad \min_{\Lambda^*} \max_{\Lambda} \Big\{ \sum_i \min_p \Big\{ p_i q_i - S_i(p_i/\Lambda^*, \Lambda) \Big\} \; - \; \log \Lambda^* \; + \; G(\Lambda) \Big\} \\
&= \quad \min_{\Lambda^*} \max_{\Lambda} \Big\{ \sum_i u_i(q_i \Lambda^*, \Lambda) \; - \; \log \Lambda^* \; + \; G(\Lambda) \Big\}
\end{aligned}$$

E Non-homothetic demand as a function of normalized prices

In this setting, it is useful to consider all objects as functions of the log of normalized income, $x_i = \log(p_i/w)$, for each good i, and express everything in terms of x instead of p. As such, we focus on attention on expenditure shares satisfying:

$$\frac{p_i q_i}{w} \;=\; W_i(x_i, \Lambda_1(x), ..., \Lambda_K(x)) \tag{62}$$

where x refers to the full vector of log normalized prices $\{\log(p_i/w)\}$. Again, we focus on demand from a rational consumer, i.e. maximizing a quasi-concave utility U, or equivalently a quasi-convex indirect utility V. Expressing V as a function of log normalized prices, Roy's identity implies that the derivative w.r.t x_i must be proportional to the expenditure share on good i, i.e. we must have:

$$\frac{\partial V}{\partial x_i}(x) \;=\; \mu(x) W_i(x_i, \Lambda(x)) \tag{63}$$

where $\mu(x) \in \mathbb{R}_+$ is a positive scalar (which may vary with prices x). For most of the analysis, we assume that demand and utility are smooth, and assume away corner solutions.Note that the marginal utility of income (not taking logs) corresponds to μ divided by income.

Assumptions: [A1], [A2], [A5], [A6] as before. We alter assumption [A3]-[A4], and add [A7]:

A3. Rank of Λ. *The matrix with coefficients* $\left\{W_i, \frac{\partial W_i}{\partial \Lambda_k}\right\}$ *has full rank* $K+1$.

A4. Rank of W. *The matrix with coefficients* $\left(W_j, \frac{\partial \Lambda_k}{\partial x_j}\right)$ *has maximal rank* $K+1$*, where* K *denotes the number of aggregators), even if we drop one good i from the set of goods.*

A7. Separability in μ: $\mu(x)$ *can be expressed as a product of two scalar functions:* $\mu(x) = \chi(V(x))\lambda(\Lambda(x))$.

Without assumption [A7], we can still show that the scalar function μ defined in Equation (63) can be expressed as a function of aggregators Λ and utility V. But assumption [A7] is helpdul in obtaining an explicit indirect utility function instead of the implicit relationship described in the previous subseciton:

Proposition 7 *Demand that depends on aggregators* Λ *and satisfies all assumptions 1-7 above must take the form:*

$$W_j(x_j, \Lambda) \;=\; \frac{D_j(x_j + \Lambda^*, \Lambda')}{\lambda(\Lambda^*, \Lambda')} \tag{64}$$

where $\Lambda = \Phi(\Lambda^*, \Lambda')$ *and* Φ *is a one-to-one re-mapping from aggregators* $\Lambda \in \mathbb{R}^K$ *to aggregators* $\Lambda^* \in \mathbb{R}$ *and* $\Lambda' \in \mathbb{R}^{K-1}$. *Moreover, up to a monotonic transformation, it is derived from an indirect utility* $V(x)$ *that satisfies:*

$$V(x) \;\;=\;\; G(\Lambda') \;+\; \int_{t=0}^{\Lambda^*} \lambda(\Phi(t, \Lambda'))\, dt \;-\; \sum_j \int_{t=0}^{x_j + \Lambda^*} D_j(t, \Lambda')\, dt \tag{65}$$

for some real function $G(\Lambda')$*, and where the aggregators* (Λ^*, Λ') *are such that the partial derivatives of the RHS in* (Λ^*, Λ') *are null.*

Rationalization and utility function. Conversely, we can obtain sufficient conditions under which the V function defined above is a well-behaved indirect utility function, in particular to ensure quasi-convexity. As in Lemma 4, it is convenient to disentangle convex and concave cases, and both cases are appropriate as they preserve convexity.

Proposition 8 *Suppose that indirect utility* $V(x)$ *is defined by:*

$$V(x) \;\;=\;\; G(\Lambda) \;+\; \int_{t=0}^{\Lambda^*} \lambda(t, \Lambda)\, dt \;-\; \sum_j \int_{t=0}^{x_j + \Lambda^*} D_j(t, \Lambda)\, dt \tag{66}$$

for some real functions $D_i(x_i, \Lambda)$*,* $G(\Lambda)$*,* $\lambda(\Lambda^*, \Lambda)$*, where the right-hand side is either concave in* (Λ, x) *or convex in* Λ*, and where it is either concave or convex in* Λ^* *(given* Λ*). Assume also that aggregators* $\Lambda(x)$ *and* $\Lambda^*(x)$ *are such that the RHS has a zero derivative in* Λ *and* Λ^* *is null. Then indirect utility* V *is quasi-convex in normalized prices, strictly decreasing in each price, and leads to the demand function* W_i *as in expression (64).*

The functional form taken by direct utility is then very similar. The same conditions as above allow us express prices and marginal utilities as functions of quantities q and quantity aggregators in the same fashion as above. Note that symmetry obtained here is not typical of less general forms of separability; in particular, indirect additive separability is not equivalent to indirect additive separability. Here direct utility can be expressed as:

$$U \;=\; \sum_i \int_0^{\log q_i + \widetilde{\Lambda}} \widetilde{D}_i(t, \Lambda) dt \;-\; \int_0^{\widetilde{\Lambda}} \widetilde{\lambda}(t, \Lambda)\, dt \;+\; G(\Lambda) \tag{67}$$

with aggregators $\widetilde{\Lambda}$ that plays a similar role as Λ^* previously. The primitive of $\widetilde{D}_i$ is again the Fenchel conjugate of the primitive of D_i, and $\widetilde{\lambda}$ is defined such that $\widetilde{\lambda}(t', \Lambda) = \lambda(t, \Lambda)$ for $t' = t - \log \lambda(t, \Lambda)$.[16]

Generalization In Appendix, we consider an even more general form of demand with aggregators allows for indirect utility V as one of its aggregators, yet without imposing homogeneity for other aggregators. This combines cases developed in the previous two sections.

[16] When that inverse does not exist, a similar expression can still be obtained by having λ appear in the $\widetilde{D}_i$ function.