# ARE 202, T. Fally - Problem set 4 – Returns to factors

The goal of this problem set is to help you become familiar with dealing with general equilibrium involving several markets at the same time. You need to examine the relative demand for factors, the relative price of factors, the relative supply and demand for goods, their relative price, combined with income effects in consumption.

## Return to capital and consumption patterns

Suppose that we have two factors of production: capital and labor. Denote by $r$ the return to capital and by $w$ the wage of workers. Suppose that we have two goods (1 and 2) that are produced with constant returns to scale under perfect competition. Let us denote by $a_{iK}(r,w)$ and $a_{iL}(r,w)$ the unit factor requirements for capital and labor respectively in sector $i$. We further assume that good 1 is always more intensive in capital than good 2 (no factor intensity reversal):

$$\frac{a_{1K}(r,w)}{a_{1L}(r,w)} > \frac{a_{2K}(r,w)}{a_{2L}(r,w)} \quad \text{for any} \quad (w,r)$$

On the demand side, we assume that we have a representative consumer whose income combines the income from labor and from capital.

A. For this question, suppose that we have homothetic preferences. Suppose that we have an increase in productivity, which means that all factor requirements (capital and labor) are multiplied by the same scalar $\lambda < 1$ in all sector. Show that the relative return to capital $\frac{r}{w}$ does not change in general equilibrium.

B. (more difficult) Now, suppose that we have non-homothetic preferences, assuming that good 1 is more income-elastic than good 2 (at any price and income levels). Yet, assume that good 2 is not a Giffen good. Again, suppose that we experience an increase in productivity, which means that all factor requirements (capital and labor) are multiplied by the same scalar $\lambda < 1$ in all sector. Show that the return to capital $\frac{r}{w}$ is higher in the new equilibrium.

## Questions providing intermediate steps and guidance

For both questions, the focus is on the relative cost of capital $\frac{r}{w}$. Denote relative production by $y = \frac{Y_1}{Y_2}$ and the ratio of prices $p = \frac{p_1}{p_2}$. Also denote income relative to the price of good 1: $e = E/p_1$. Questions 1 through 5 below apply to both parts A and B.

1. Why do K and labor unit requirements $a_{iK}$ and $a_{iL}$ only depend on relative factor price $r/w$?

2. Regardless of the type of demand (part A or B), show that factor market clearing requires:

$$\frac{K}{L} = \frac{a_{1K}(r/w) \cdot y + a_{2K}(r/w)}{a_{1L}(r/w) \cdot y + a_{2L}(r/w)}$$

3. Since relative factor supply is fixed, this equality describes a relationship between factor prices $r/w$ and relative supply $y$ and implicitly defines a function $s$ such that:

$$r/w = s(y) \tag{1}$$

   Show that function $s(y)$ is increasing in $y$, i.e. that $r/w$ increases with the relative size of industry 1.

4. With perfect competition and constant returns to scale, prices are given by unit costs. Taking the ratio of costs, show that the relative $p = p_1/p_2$ price of good 1 can be written as a function of the relative cost of capital $r/w$:

$$p = c(r/w) \tag{2}$$

5. Show that the function $c(r/w)$ defined above is increasing in $r/w$.

6. Next, you can examine what determines relative demand $y$ for good 1 relative to good 2. For part A) with homothetic preferences, explain why the relative demand for good 1 is an decreasing function of the relative price $p$:

$$y = d_A(p) \tag{3}$$

7. For part B, relative demand $y$ also depend on income. I find it practical to define relative income as the ratio of nominal income to the price of good 1: $e = \frac{E}{p_1}$. Explain why we can express relative demand is a function of just relative prices and relative income:

$$y = d_B(p, e) \tag{4}$$

   Does it necessarily decrease with the relative price $p$ and increase with relative income $e = E/p_1$?

8. Finally, since consumer income is the sum of income from labor and income from capital, show that relative income $e = E/p_1$ can be expressed as a function of $\lambda$ and $r/w$:

$$e = e(\lambda, r/w) \tag{5}$$

   [Note: use $p_1 = \lambda c_1(r, w)$]. How does $e(\lambda, r/w)$ depend on $\lambda$?

9. The previous questions provide a set of four equations that characterize the equilibrium in all four markets. In general equilibrium, how does the productivity shifter $\lambda$ affect $r/w$ in parts A and B, depending on whether demand is homothetic?

**General hints and tips**

- If you are stuck or lost, you can start by imposing a specific functional form, e.g. Cobb-Douglas production functions on the supply side and Cobb-Douglas (Q1) or Stone-Geary or CREI (Q2) demand systems on the consumption side. Please keep that tip in mind for your future research. Do not try to start big when you build a model, first make sure you get the intuition with simple cases before trying to be more general. It's easy to get lost otherwise.
- As mentioned in the lectures, in general equilibrium with constant returns to scale in production, it is often more practical to examine equilibrium on factor markets in relative terms: relative cost of capital $r/w$, relative demand for capital $K/L$, relative production, relative cost, etc.
- To answer question 9, I find it convenient to summarize the equilibrium with just two curves (e.g. "demand" and "supply" curves), each combining some of the equations above.