ARE 202, T. Fally – Problem set 1 – Additive preferences

Part A. Consider utility functions of the form:

$$U = \sum_i u_i(x_i)$$

where x_i denotes the consumption of good i. Let's denote by w the consumer's income. We will also denote by λ the Lagrange multiplier. A utility function is said to be directly additive (or separable) if it can be written as above, up to a monotonic transformation. Also, we will assume that:

$$u_i'(x) > 0 \quad \text{and} \quad u_i''(x) < 0$$

In this problem set, we also assume that marginal utility is unbounded: $\lim_{x \to 0} u_i'(x) = +\infty$.

Questions:

1. Is Cobb-Douglas a separable utility function? what about CES? Stone-Geary?
2. Let us denote by $D_i = u_i'^{-1}$ the inverse function of marginal utility u_i'. Is it well defined? Is D_i an increasing function? Express demand for good i using D_i, the Lagrange multiplier λ and prices p_i.
3. Prove that the Lagrange multiplier λ decreases with income.
4. Can good i be an inferior good? Can good i be a Giffen good? [prove or give a counter-example]
5. Show the following relationship between the own price elasticity $\frac{\partial \log x_i}{\partial \log p_i}$ (of the Marshallian demand), the income elasticity $\frac{\partial \log x_i}{\partial \log w}$, and the elasicity of λ w.r.t. p_i and λ:
$$\frac{\partial \log x_i}{\partial \log p_i} \cdot \frac{\partial \log \lambda}{\partial \log w} = \frac{\partial \log x_i}{\partial \log w} \left[1 + \frac{\partial \log \lambda}{\partial \log p_i}\right]$$
6. When there is a large number of goods, explain why $\frac{\partial \log \lambda}{\partial \log p_i} \approx 0$ [no math required].
7. Using the previous question, we can conclude that:
$$\frac{\partial \log x_i}{\partial \log p_i} \cdot \frac{\partial \log \lambda}{\partial \log w} \approx \frac{\partial \log x_i}{\partial \log w}$$
which means that the own price elasticity of good i is proportional to the income elasticity of good i when each good has a small market share. This point was made by Pigou. From your own experience/preferences, can you provide casual examples where this prediction does *not* seem to hold?

Part B. To avoid restrictions implied by Pigou's law, an alternative is to define utility implicitly (implicitly-additive utility). For instance, suppose that utility U is the solution of:

$$\sum_i A_i^{\frac{1}{\sigma_i}} \left(\frac{x_i}{U}\right)^{\frac{\sigma_i - 1}{\sigma_i}} = 1$$

where A_i and σ_i are positive parameters, with $\sigma_i > 1$ for all i.

Questions:

8. Is that utility function increasing in each x_i? quasi-concave? strictly quasi-concave?

9. What is the own-price elasticity of good i, assuming that it has a negligible expenditure share?

10. What is the income elasticity of good i, assuming that it has a negligible expenditure share?