The B.E. Journal of Theoretical Economics

Topics

Volume 12, *Issue* 1 2012 *Article* 9

Strict Concavity of the Value Function for a Family of Dynamic Accumulation Models

Eugenio S.A. Bobenrieth* Juan R.A. Bobenrieth†
Brian D. Wright‡

*Pontificia Universidad Católica de Chile, ebobenrieth@uc.cl
†Universidad del Bío-Bío, jbobenri@ubiobio.cl
‡University of California, Berkeley, bwright@berkeley.edu

Recommended Citation
Eugenio S.A. Bobenrieth, Juan R.A. Bobenrieth, and Brian D. Wright (2012) “Strict Concavity of the Value Function for a Family of Dynamic Accumulation Models,” *The B.E. Journal of Theoretical Economics*: Vol. 12: Iss. 1 (Topics), Article 9.

DOI: 10.1515/1935-1704.1838

Strict Concavity of the Value Function for a Family of Dynamic Accumulation Models*

Eugenio S.A. Bobenrieth, Juan R.A. Bobenrieth, and Brian D. Wright

Abstract

We prove strict concavity of the value function for liquidity constrained dynamic accumulation models without adopting at least one of the following restrictive assumptions: zero response of productive effort, bounded marginal value of accumulated balances, or strictly convex cost of holding accumulated balances. Thus we extend well known theoretical results to more general models of saving with liquidity constraints and of commodity storage with non-negativity constraints on stocks. Our results provide a foundation for estimation of a homogeneous markovian process for consumption in models of saving, or for price in commodity storage models, under more realistic assumptions.

KEYWORDS: consumption, savings, strict concavity, value function

*Work on this paper was supported by the Energy Biosciences Initiative, and by CONICYT/Fondo Nacional de Desarrollo Científico y Tecnológico (FONDECYT) Projects 1090017 and 1050562. Eugenio Bobenrieth's research for this paper was done partially when he was a professor at Universidad de Concepción, Chile. Eugenio Bobenrieth acknowledges partial financial support from Grupo Security through Finance UC, and from Project NS 100046 of the Iniciativa Científica Milenio of the Ministerio de Economía, Fomento y Turismo, Chile. We are grateful to two anonymous referees for useful questions and suggestions.

Eugenio S.A. Bobenrieth: Departamento de Economía Agraria, Instituto de Economía, and Finance UC, Pontificia Universidad Católica de Chile.
Juan R.A. Bobenrieth: Departamento de Matemática, Universidad del Bío-Bío.
Brian D. Wright: Department of Agricultural and Resource Economics, University of California, Berkeley.

1 Introduction

In this paper we prove the strict concavity of the value function for a class of stationary surplus maximization models in which the accumulated balance of savings, or stocks of a consumable commodity, is non-negative and may grow or depreciate, and its marginal value may be unbounded. Supply of effort may be economically responsive, and the income or supply disturbance, if any, is i.i.d. For this class of accumulation models it is well known that the value function is weakly concave. However strict concavity of the value function is required for the Markov consumption process to be time-homogeneous (given an initial value for available resources). Time-homogeneity of the consumption process is necessary for application of standard results of ergodic theory. This is important in laying the foundation for asymptotic results that are currently used in econometric implementations.

In a commodity storage model, for example, strict concavity of the value function is necessary to ensure that the market demand function is well defined. This is important to derive standard theoretical results, in particular to establish that the price process has a unique invariant distribution which is a global attractor (see Scheinkman and Schechtman, 1983, p. 435, and Bobenrieth et al., 2002, p. 1217).

We have not been able to find a general proof of strict concavity for this model in the literature.

Strict concavity is established only in special cases of the model adopting restrictive assumptions. For example Deaton and Laroque (1992), which establishes the foundation for the discussion of saving with liquidity constraints in Deaton (1991), and for the econometric applications in Deaton and Laroque (1995) and Deaton and Laroque (1996), assumes that supply of effort is unresponsive and that marginal direct utility is bounded. In Bobenrieth et al. (2002), Scheinkman and Schechtman (1983), and Stokey et al. (1989, section 10.5, pp. 297-300), the restriction is that the cost of holding accumulated balances is strictly convex. This assumption is clearly inappropriate for models of saving, and does not have empirical support with respect to commodity storage.[1] Indeed the pioneering storage model of Gustafson (1958), and many subsequent dynamic accumulation models, assume constant marginal storage cost.

The family of models we address include the commodity storage model with constant consumption demand elasticity and possibly responsive supply in Wright and Williams (1982), and the model of saving with martingale

[1]Reports usually imply that marginal storage cost is approximately constant over a large range of stocks, perhaps increasing as the usual storage facilities approach their capacity (Paul, 1970, UNCTAD, 1975).

discounted marginal value in Bobenrieth et al. (2011) where the transition probability for marginal value is well defined if marginal value is strictly decreasing in total available resources.

A particular case of the model in this paper is the standard non-stochastic one-sector model of optimal growth with linear technology when capital is the maintainable capital stock, as described in Stokey et al. (1989, section 5.1, p. 104). We note that the argument for strict concavity offered by Stokey et al. (1989, exercise 5.1 c., p. 104) and the solution proposed by Irigoyen et al. (2002, p. 53) are both wrong, since Assumption 4.7 of Stokey et al. (1989, p. 80) is not satisfied in this case.

The remainder of the paper is organized as follows. In section 2 we present the model and prove that supply of effort is non-increasing in available resources. In section 3 we present and prove our main result, the strict concavity of the value function. Section 4 presents a brief conclusion.

2 The Model

We address a standard model of maximization of expected surplus, with liquidity constraints, and with (possibly) responsive labor supply. Time is discrete. Income or supply is subject to one common exogenous i.i.d. multiplicative disturbance $\omega \geq 0$, with (possibly unbounded) support S. Given the accumulated balance carried forward, $x \geq 0$, and effort $\lambda \geq 0$, the amount available in the next period is $z' \equiv Rx + \omega'\lambda$, where $R > 0$ is a depreciation or appreciation factor that is assumed to be known with certainty, and ω' is the disturbance in the next period. The discount factor is δ, $0 < \delta < 1$. We assume that $\delta R < 1$. If λ is economically responsive, cost of effort is given by a function $g : \mathbb{R}_+ \to \mathbb{R}_+$, with $g(0) = 0$, $g'(0) = 0$, and $g'(\lambda) > 0$, $g''(\lambda) > 0$ for all $\lambda > 0$. Storage cost is given by a function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$, with $\phi(0) = 0$, and $\phi'(x) \geq 0$, $\phi''(x) \geq 0$ for all $x \geq 0$.

The utility of consumption is $U : \mathbb{R}_+ \to \mathbb{R}$. U is continuous, once continuously differentiable, strictly increasing and strictly concave. We assume that U has a finite upper bound. U' need not be bounded. Let $f \equiv U'$.

For available resources $z \geq 0$, the Bellman equation for the surplus problem is:

$$\nu(z) = \max_{x,\lambda}\{U(z - x) - \phi(x) - g(\lambda) + \delta E[\nu(z')]\}, \quad \text{subject to} \tag{1}$$

$$z' = Rx + \omega'\lambda,$$

$$0 \leq x \leq z, \quad \lambda \geq 0,$$

where $E[.]$ denotes the expectation with respect to next period's disturbance ω'.

By standard results (see for example Stokey et al., 1989), ν exists, is unique, bounded, continuous, strictly increasing and weakly concave. For a given $z \geq 0$, the objective function in (1) is strictly concave in (x, λ), implying that the optimal policy functions $x(z)$ and $\lambda(z)$ are single-valued. Continuity of $x(z)$ and $\lambda(z)$ follows from continuity of the objective function and continuity of the restrictions, in (1). Consumption and marginal value are given by the functions $c(z) \equiv z - x(z),\ p(z) \equiv f(z - x(z))$.

We prove that the marginal value, ν', exists and $\nu'(z) = p(z),\ \forall z \geq 0$ (Proposition A.2, Appendix). The policy functions x and λ satisfy the Euler conditions:

$$p(z) + \phi'(x(z)) \geq \delta R E[p(Rx(z) + \omega'\lambda(z))], \quad \text{with equality if } \ x(z) > 0, \tag{2}$$

$$g'(\lambda(z)) \geq \delta E[\omega' p(Rx(z) + \omega'\lambda(z))], \quad \text{with equality if } \ \lambda(z) > 0. \tag{3}$$

The Euler conditions imply the following preliminary result, which is based in the fact that an increase in available resources z does not increase $E[\omega'\nu'(z')]$; thus, by convexity of g, does not increase the incentive for effort λ. The proof is related to the proof of Theorem 2 (b) of Scheinkman and Schechtman (1983, p. 432), and to the argument in Stokey et al. (1989, Section 10.5, p. 300).

Lemma. *λ is non-increasing in* $[0, \infty)$.

Proof of the Lemma. If $S = \{0\}$, (3) implies $\lambda(z) \equiv 0$. If $S \neq \{0\}$, by contradiction assume that there exist $0 \leq z^{(0)} < z^{(1)}$ such that $\lambda(z^{(0)}) < \lambda(z^{(1)})$.

Let $\gamma(\omega') \equiv p(Rx(z^{(0)}) + \omega'\lambda(z^{(0)})) - p(Rx(z^{(1)}) + \omega'\lambda(z^{(1)}))$.

Since $g'(0) = 0,\ \lambda(z) > 0,\ \forall\ z \geq 0$. By (3), and the fact that g is strictly convex, $E[\omega'\gamma(\omega')] < 0$. This implies, given p is non-increasing (equivalently, ν, is weakly concave) that $x(z^{(0)}) > x(z^{(1)})$. We now show that $E[\gamma(\omega')] > 0$. Since $x(z^{(0)}) > x(z^{(1)}) \geq 0$, by (2) we have:

$$p(z^{(0)}) + \phi'(x(z^{(0)})) = \delta R E[p(Rx(z^{(0)}) + \omega'\lambda(z^{(0)}))], \tag{4}$$

and

$$p(z^{(1)}) + \phi'(x(z^{(1)})) \geq \delta R E[p(Rx(z^{(1)}) + \omega'\lambda(z^{(1)}))]. \tag{5}$$

Given $z^{(0)} < z^{(1)}$ and $x(z^{(0)}) > x(z^{(1)})$, it follows that $c(z^{(0)}) < c(z^{(1)})$ and therefore $p(z^{(0)}) > p(z^{(1)})$. Subtracting (5) from (4), we obtain $\delta R E[\gamma(\omega')] > \phi'(x(z^{(0)})) - \phi'(x(z^{(1)})) \geq 0$, thus $E[\gamma(\omega')] > 0$.

If $\gamma(\hat{\omega}) > 0$, then $\gamma(\omega') \geq 0$, $\forall\ \omega' \geq \hat{\omega}$, and if $\gamma(\tilde{\omega}) < 0$, then $\gamma(\omega') \leq 0$, $\forall\ \omega' \leq \tilde{\omega}$. Therefore there exists $\omega^* \equiv \sup\{\omega' : \gamma(\omega') < 0\}$. In the expectation $E[\omega'\gamma(\omega')] < 0$, all negative values of $\gamma(\omega')$ are weighted by values ω', $0 \leq \omega' \leq \omega^*$, and all positive values of $\gamma(\omega')$ have weights $\omega' \geq \omega^*$, leading to a contradiction given the fact that $E[\gamma(\omega')] > 0$.

Q.E.D.

3 The Theorem

Theorem. *The value function ν is strictly concave.*

The result follows from the Euler conditions. The impatience assumption $\delta R < 1$ is a central element of the proof. For convenience in the exposition, the proof of the Theorem is done separately for the deterministic and the stochastic cases. The proof for the deterministic case extends the solution presented in the discussion of a consumption-savings model in section 5.17 of Stokey et al. (1989, pp. 126-128) to the case of responsive labor supply and possibly unbounded marginal value, the main idea is to construct a strictly concave function that satisfies the Bellman equation. The result follows from the uniqueness of the value function. In the proof for the stochastic case we show that the existence of an interval in which ν' is constant leads to a contradiction.

Proof of the Theorem for the deterministic case: We first consider the case where $U'(0) < \infty$ or $\omega \neq 0$. We assume $\omega \equiv \underline{\omega} \geq 0$, a constant. We construct a function $\nu^{(\underline{\omega})} : \mathbb{R}_+ \to \mathbb{R}$ that is strictly concave, and we prove that $\nu^{(\underline{\omega})}$ satisfies the Bellman equation.
Define $\{a^{(0)}, a^{(1)}, \cdots\}$, $\{x^{(\underline{\omega})}(a^{(0)}), x^{(\underline{\omega})}(a^{(1)}), \cdots\}$ and $\{\lambda^{(\underline{\omega})}(a^{(1)}), \lambda^{(\underline{\omega})}(a^{(2)}), ..\}$ as the sequences of numbers that satisfy:

$$
\begin{aligned}
&Rx^{(\underline{\omega})}(a^{(j+1)}) + \underline{\omega}\,\lambda^{(\underline{\omega})}(a^{(j+1)}) = a^{(j)},\\
&g'(\lambda^{(\underline{\omega})}(a^{(j+1)})) = \delta\,\underline{\omega}\, f(a^{(j)} - x^{(\underline{\omega})}(a^{(j)})), \quad \text{and}\\
&f(a^{(j+1)} - x^{(\underline{\omega})}(a^{(j+1)})) + \phi'(x^{(\underline{\omega})}(a^{(j+1)})) = \delta R f(a^{(j)} - x^{(\underline{\omega})}(a^{(j)})),\\
&\text{with } \ x^{(\underline{\omega})}(a^{(0)}) = x^{(\underline{\omega})}(a^{(1)}) \equiv 0.
\end{aligned}
$$

Note that $0 \leq a^{(0)} < a^{(1)} < a^{(2)} < \cdots$, and $a^{(j)} \to \infty$ (as $j \to \infty$).
We now construct the functions $x^{(\underline{\omega})} : \mathbb{R}_+ \to \mathbb{R}_+$, $\quad \lambda^{(\underline{\omega})} : \mathbb{R}_+ \to \mathbb{R}_+$. In the interval $[0, a^{(1)}]$, we define $x^{(\underline{\omega})} \equiv 0, \lambda^{(\underline{\omega})} \equiv \lambda^{(\underline{\omega})}(a^{(1)})$. Inductively for $j = 0, 1, 2, \cdots$, and for a given $z' \in [a^{(j)}, a^{(j+1)}]$, there exist unique numbers z, x, λ such that:

$$\begin{cases} Rx + \underline{\omega}\,\lambda = z' \\ g'(\lambda) = \delta\,\underline{\omega}\, f(z' - x^{(\underline{\omega})}(z')) \\ f(z-x) + \phi'(x) = \delta R f(z' - x^{(\underline{\omega})}(z')). \end{cases} \tag{6}$$

As z' traverses $[a^{(j)}, a^{(j+1)}]$, z traverses $[a^{(j+1)}, a^{(j+2)}]$, and the correspondence $z' \mapsto z$ is one to one. The solution (z, x, λ) of (6) is denoted by $(z, x^{(\underline{\omega})}(z), \lambda^{(\underline{\omega})}(z))$. The functions $x^{(\underline{\omega})}(z)$ and $\lambda^{(\underline{\omega})}(z)$ depend continuously on $z \in [a^{(j+1)}, a^{(j+2)}]$, and $c^{(\underline{\omega})}(z) \equiv z - x^{(\underline{\omega})}(z)$ is strictly increasing in $z \in [a^{(j+1)}, a^{(j+2)}]$.
We construct $\nu^{(\underline{\omega})} : \mathbb{R}_+ \to \mathbb{R}$ as $\nu^{(\underline{\omega})}(z) \equiv U(z - x^{(\underline{\omega})}(z)) - \phi(x^{(\underline{\omega})}(z)) - g(\lambda^{(\underline{\omega})}(z)) + \delta\nu^{(\underline{\omega})}(Rx^{(\underline{\omega})}(z) + \underline{\omega}\lambda^{(\underline{\omega})}(z))$. Using the definition of $x^{(\underline{\omega})}(z), \lambda^{(\underline{\omega})}(z)$, it is straightforward to check that $(\nu^{(\underline{\omega})})'(z) = U'(z - x^{(\underline{\omega})}(z)),\ \forall\, z \geq 0$ (the Benveniste-Scheinkman derivative, Benveniste and Scheinkman, 1979). Since U' is strictly decreasing and $c^{(\underline{\omega})}(z) \equiv z - x^{(\underline{\omega})}(z)$ is strictly increasing, $\nu^{(\underline{\omega})}$ is strictly concave.
Finally, $\nu^{(\underline{\omega})}$ satisfies the Bellman equation. The argmax of the optimization problem:

$$\nu^{(\underline{\omega})}(z) = \max_{\substack{0 \leq x \leq z \\ 0 \leq \lambda}} \{U(z-x) - \phi(x) - g(\lambda) + \delta\nu^{(\underline{\omega})}(Rx + \underline{\omega}\lambda)\}$$

is $x^{(\underline{\omega})}(z), \lambda^{(\underline{\omega})}(z)$.

For the case where $U'(0) = \infty$ and $\omega \equiv 0$, consider:

$$\varphi(q) \equiv D(q) + \sum_{j=1}^{\infty} \frac{1}{R^j} D(p(z_j)),$$

where $D = f^{-1}$, $z_j = Rx(z_{j-1})$ (for $j \in \mathbb{N}$), and $p(z_0) = q$. Note that $\varphi : (0,\infty) \to (0,\infty)$ is a bijection and $p(\varphi(q)) = q$, for all $q \in (0,\infty)$. Therefore, the price function p is strictly decreasing. *Q.E.D.*

Proof of the Theorem for the stochastic case: We assume that S is not a singleton. Since $p(z) = \nu'(z)\ \forall\, z \geq 0$ and ν is concave, p is non-increasing in $[0,\infty)$. Proof by contradiction proceeds by supposing that there exists a first interval $I_1 = [z^{(0)}, z^{(1)}],\ z^{(0)} < z^{(1)}$, where p is constant. By Proposition A.1 (Appendix), $z^{(0)} > 0$. Furthermore, the fact that $\lim_{z\to\infty} p(z) = 0$ (since ν is bounded and $\nu' = p$) implies $z^{(1)} < \infty$.
Since λ is non-increasing in $[0,\infty)$, we consider two possible cases:
Case 1: $\lambda(z^{(0)}) > \lambda(z^{(1)})$. Let

$$\zeta(\omega') \equiv p(R(z^{(1)} - c) + \omega'\lambda(z^{(1)})) - p(R(z^{(0)} - c) + \omega'\lambda(z^{(0)})),$$

where c is the constant value of consumption on I_1. By the argument in the proof of the Lemma (for the case $S \neq \{0\}$), we conclude that $E[\omega'\zeta(\omega')] < 0$ and $E[\zeta(\omega')] \geq 0$, a contradiction.
Case 2: $\lambda(z^{(0)}) = \lambda(z^{(1)})$. Therefore λ is constant on I_1. By (2) and the fact that p is non-increasing,

$$E[p(R(z^{(0)} - c) + \omega'\lambda)] = E[p(R(z^{(1)} - c) + \omega'\lambda)],$$

where λ is the constant value of effort on I_1, and c is the constant value of consumption on I_1. If S is unbounded, we conclude that p is a positive constant in $[R(z^{(0)} - c) + \underline{\omega}\lambda, \infty)$, where $\underline{\omega} \equiv \inf S$, a contradiction to the boundedness of utility. If S is bounded, p is constant in $[R(z^{(0)} - c) + \underline{\omega}\lambda, R(z^{(1)} - c) + \overline{\omega}\lambda]$, where $\overline{\omega} \equiv \sup S$. By (2) and the assumption $\delta R < 1$, we conclude that $R(z^{(1)} - c) + \overline{\omega}\lambda < z^{(0)}$. Therefore $[R(z^{(0)} - c) + \underline{\omega}\lambda, R(z^{(1)} - c) + \overline{\omega}\lambda]$ is strictly to the left of the interval I_1. Hence, there is an interval of positive length $I_2 = [z^{(2)}, z^{(3)}]$, $I_2 \supseteq [R(z^{(0)} - c) + \underline{\omega}\lambda, R(z^{(1)} - c) + \overline{\omega}\lambda]$, strictly to the left of the interval I_1, where p is constant, a contradiction. *Q.E.D.*

4 Conclusion

We prove strict concavity of the value function for liquidity constrained dynamic accumulation models without adopting at least one of the following restrictive assumptions: zero response of productive effort, bounded marginal value of accumulated balances, or strictly convex cost of holding accumulated balances. Strict concavity implies strict monotonicity of the consumption function, and of the marginal value function. Thus we extend well known theoretical results to more general models of saving with liquidity constraints and of commodity storage with non-negativity constraints on stocks. Given the time-homogeneous Markov process for the accumulated balance, $z' = Rx(z) + \omega'\lambda(z)$, strict concavity of ν is required for time-homogeneity of the transition probability for the consumption, and for the marginal value or price, processes. Our results provide a foundation for estimation of a homogeneous markovian process for consumption in models of saving, and for price in commodity storage models, under more realistic assumptions.

Appendix

Proposition A.1. $c(z) > 0, \ \forall \ z > 0.$

The proof of Proposition A.1 is immediate in the case $U'(0) = \infty$, based on the concavity of ν. If $U'(0)$ is finite, the basic idea is to use a limit argument on the corresponding finite horizon problems.

Proof of Proposition A.1.

Case 1: $U'(0) = \infty$. If $c(z^{(0)}) = 0$ for some $z^{(0)} > 0$, then $\nu'_+(z^{(0)}) = \infty$, a contradiction to the fact that ν is concave.

Case 2: $U'(0) < \infty$. Consider the following sequences of functions, for the corresponding finite horizon problems:

$$\nu_0 \equiv 0, \qquad \nu_t(z) = \max_{\substack{0\le x\le z\\ 0\le\lambda}} \{U(z-x) - \phi(x) - g(\lambda) + \delta E[\nu_{t-1}(Rx + \omega'\lambda)]\},$$

$$(x_t(z), \lambda_t(z)) \;=\; \arg\max_{\substack{0\le x\le z\\ 0\le\lambda}} \{U(z-x) - \phi(x) - g(\lambda) + \delta E[\nu_{t-1}(Rx + \omega'\lambda)]\},$$

$t = 1, 2, \cdots$.

Note that ν_t is strictly concave $\forall\, t$. By standard results, $\{\nu_t\}_{t\ge 0}$ converges uniformly to the solution ν of the Bellman equation (1), and $\{(x_t(z), \lambda_t(z))\}_{t\in\mathbb{N}}$ converges pointwise to $(x(z), \lambda(z))$. Denote by $\{c_t(z)\}_{t\in\mathbb{N}}$ the sequence of functions $c_t(z) \equiv z - x_t(z), \;\; \forall\, t \in \mathbb{N}$.

Choose $\underline{\omega} = \inf S$, to be the constant value of ω in the deterministic case. For this case, denote by $\{\nu_t^{(\underline{\omega})}\}_{t\ge 0}$ the sequence of functions:

$$\nu_0^{(\underline{\omega})} \equiv 0, \qquad \nu_t^{(\underline{\omega})}(z) = \max_{\substack{0\le x\le z\\ 0\le\lambda}} \{U(z-x) - \phi(x) - g(\lambda) + \delta \nu_{t-1}^{(\underline{\omega})}(Rx + \underline{\omega}\lambda)\},$$

and denote by $\{c_t^{(\underline{\omega})}(z)\}_{t\in\mathbb{N}}$ the sequence of functions $c_t^{(\underline{\omega})}(z) \equiv z - x_t^{(\underline{\omega})}(z)$, where

$$(x_t^{(\underline{\omega})}(z), \lambda_t^{(\underline{\omega})}(z)) \;=\; \arg\max_{\substack{0\le x\le z\\ 0\le\lambda}} \{U(z-x) - \phi(x) - g(\lambda) + \delta \nu_{t-1}^{(\underline{\omega})}(Rx + \underline{\omega}\lambda)\},$$

$t = 1, 2, \cdots$.

Let $c^{(\underline{\omega})}(z)$ be the pointwise limit of the sequence $\{c_t^{(\underline{\omega})}(z)\}_{t\in\mathbb{N}}$. The following result establishes that $c^{(\underline{\omega})}(z)$ is a lower bound for the consumption function $c(z) \equiv z - x(z)$.

Statement: $c(z) \ge c^{(\underline{\omega})}(z), \; \forall\, z \ge 0$.

Proof of the Statement: It suffices to prove that

$$c_t(z) \ge c_t^{(\underline{\omega})}(z), \quad \forall\, z \ge 0, \quad \forall\, t \in \mathbb{N}.^{2}$$

Proceeding by induction:

[2] This result is stated in Theorem 4.1 of Schechtman (1973, p. 27), for a model with unresponsive labor supply.

For $t = 1: \quad c_1(z) = z = c_1^{(\underline{\omega})}(z), \ \forall \ z \geq 0.$

Assume $c_{t-1}(z) \geq c_{t-1}^{(\underline{\omega})}(z), \quad \forall \ z \geq 0$ (and therefore $\nu'_{t-1}(z) = U'(c_{t-1}(z)) \leq U'(c_{t-1}^{(\underline{\omega})}(z)) = (\nu_{t-1}^{(\underline{\omega})})'(z), \quad \forall \ z \geq 0$).
To prove that $c_t(z) \geq c_t^{(\underline{\omega})}(z)$, consider the non-trivial case of $z > 0$ and $c_t(z) < z$. By contradiction, assume that $c_t(z) < c_t^{(\underline{\omega})}(z)$ (and therefore $x_t(z) > x_t^{(\underline{\omega})}(z)$). By the optimality conditions for $x_t(z)$ and $x_t^{(\underline{\omega})}(z)$:

$$E[\nu'_{t-1}(Rx_t(z) + \omega'\lambda_t(z))] \ > \ (\nu_{t-1}^{(\underline{\omega})})'(Rx_t^{(\underline{\omega})}(z) + \underline{\omega}\,\lambda_t^{(\underline{\omega})}(z)), \tag{7}$$

thus

$$E[\omega'\nu'_{t-1}(Rx_t(z) + \omega'\lambda_t(z))] \ > \ \underline{\omega}\,(\nu_{t-1}^{(\underline{\omega})})'(Rx_t^{(\underline{\omega})}(z) + \underline{\omega}\,\lambda_t^{(\underline{\omega})}(z)).$$

Since ν_{t-1} is strictly concave and $\nu'_{t-1} \leq (\nu_{t-1}^{(\underline{\omega})})'$, (7) implies that $\lambda_t(z) < \lambda_t^{(\underline{\omega})}(z)$. The optimality conditions for $\lambda_t(z)$ and $\lambda_t^{(\underline{\omega})}(z)$ imply that:

$$\begin{aligned} \delta E[\omega'\nu'_{t-1}(Rx_t(z) + \omega'\lambda_t(z))] = g'(\lambda_t(z)) \ < \ g'(\lambda_t^{(\underline{\omega})}(z)) = \\ = \delta\underline{\omega}\,(\nu_{t-1}^{(\underline{\omega})})'(Rx_t^{(\underline{\omega})}(z) + \underline{\omega}\,\lambda_t^{(\underline{\omega})}(z)), \end{aligned}$$

a contradiction, finishing in this way the proof of the Statement.

Using the Statement, and the fact that $c^{(\underline{\omega})}(z) > 0, \ \forall \ z > 0$ we conclude the proof. *Q.E.D.*

Proposition A.2. $\nu'(z) = p(z), \ \forall \ z \geq 0.$

The proof of Proposition A.2 is based on the Benveniste-Scheinkman derivative (Benveniste and Scheinkman, 1979).

Proof of Proposition A.2. Using the fact that $c(z) > 0, \forall \ z > 0$, and using Lemma 1 of Benveniste and Scheinkman (1979, p. 728), we conclude that $\nu'(z) = p(z), \forall \ z > 0$. Using this result, the Mean Value Theorem, and the fact that $p(z) \equiv f(c(z))$ is continuous at 0, we conclude that $\nu'_+(0) = p(0)$. *Q.E.D.*

References

Benveniste, L.M., and J.A. Scheinkman (1979): "On the Differentiability of the Value Function in Dynamic Models of Economics," *Econometrica,* 47, 727-732.

Bobenrieth, E.S.A., J.R.A. Bobenrieth, and B.D. Wright (2002): "A Commodity Price Process with a Unique Continuous Invariant Distribution having Infinite Mean," *Econometrica,* 70, 1213-1219.

Bobenrieth, E.S.A., J.R.A. Bobenrieth, and B.D. Wright (2011): "The Behavior of a Martingale Discounted Commodity Price Process at Far Horizons," Working paper, Pontificia Universidad Católica and Finance UC, Universidad del Bío-Bío, University of California at Berkeley.

Deaton, A. (1991): "Saving and Liquidity Constraints," *Econometrica,* 59, 1221-1248.

Deaton, A., and G. Laroque (1992): "On the Behaviour of Commodity Prices," *Review of Economic Studies,* 59, 1-23.

Deaton, A., and G. Laroque (1995): "Estimating a Nonlinear Rational Expectations Commodity Price Model with Unobservable State Variables," *Journal of Applied Econometrics,* 10, S9-S40.

Deaton, A., and G. Laroque (1996): "Competitive Storage and Commodity Price Dynamics," *Journal of Political Economy,* 104, 896-923.

Gustafson, R. L. (1958): "Carryover Levels for Grains," *USDA Technical Bulletin 1178.* Washington, DC: United States Department of Agriculture.

Irigoyen, C., E. Rossi-Hansberg, and M.L.J. Wright (2002): *Solutions Manual for Recursive Methods in Economic Dynamics.* Harvard University Press, Cambridge.

Paul, A. B. (1970): "The Pricing of Binspace - a Contribution to the Theory of Storage," *American Journal of Agricultural Economics,* 52, 1-12.

Schechtman, J. (1973): "Some Applications of Competitive Prices to Problems of Dynamic Programming Under Uncertainty." *Ph.D. dissertation,* University of California, Berkeley.

Scheinkman, J.A., and J. Schechtman (1983): "A Simple Competitive Model with Production and Storage," *Review of Economic Studies,* 50, 1983, 427-441.

Stokey, N.L., R.E. Lucas, Jr., with E.C. Prescott (1989): *Recursive Methods in Economic Dynamics.* Harvard University Press, Cambridge.

UNCTAD (1975): "Second Progress Report on Storage Costs and Warehouse Facilities." TD/B/C.1/198, Trade and Development Board, Committee on commodities. *United Nations Conference on Trade and Development, Geneva.*

Wright, B. D., and J.C. Williams (1982): "The Economic Role of Commodity Storage." *Economic Journal,* 92, 596-614.