Appendix: Model and Comparative Statics

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1 Model and solutions

For a given firm, production takes the standard CES form $Q = F(\mathbf{X}) = A\left[\sum_{i=1}^{I}\alpha_i X_i^{\frac{\sigma-1}{\sigma}}\right]^{\nu\frac{\sigma}{\sigma-1}}$, with $\sigma > 0$ denoting the elasticity of inputs across the I inputs. Returns to scale are captured by $\nu > 0$, with $\nu = 1$ indicating constant returns to scale. The A coefficient captures Hicks-neutral total factor productivity. When $\sigma \to 1$, production converges to the Cobb-Douglas form, $F(\mathbf{X}) = A\left[\prod_{i=1}^{I} X_i^{\alpha_i}\right]^{\nu}$, with $\sum_{i=1}^{I}\alpha_i = 1$.

The partial derivatives of F with respect to input X_i are given by

$$F_i(\mathbf{X}) = \begin{cases} \nu A\left(\sum_{i=1}^{I}\alpha_i X_i^{\frac{\sigma-1}{\sigma}}\right)^{\nu\frac{\sigma}{\sigma-1}-1}\alpha_i X_i^{-1/\sigma} = \nu Q\alpha_i X_i^{-1/\sigma}\left(\sum_{i=1}^{I}\alpha_i X_i^{\frac{\sigma-1}{\sigma}}\right)^{-1} & , \sigma \neq 1 \\ \nu A\left(\prod_{i=1}^{I} X_i^{\alpha_i}\right)^{\nu}\alpha_i X_i^{-1} = \nu Q\alpha_i X_i^{-1} & , \sigma = 1 \end{cases} \tag{1}$$

while the output elasticities are given by

$$F_i(\mathbf{X})\frac{X}{Q} = \begin{cases} \nu\alpha_i X_i^{(\sigma-1)/\sigma}\left(\sum_{i=1}^{I}\alpha_i X_i^{\frac{\sigma-1}{\sigma}}\right)^{-1} & , \sigma \neq 1 \\ \nu\alpha_i & , \sigma = 1 \end{cases}$$

To ease notation in the following, let

$$\Phi = \begin{cases} \left(\sum_{i=1}^{I}\alpha_i^{\sigma}\omega_i^{1-\sigma}\right)^{\frac{1}{\sigma-1}} & , 0 < \sigma \neq 1 \\ \prod_{i=1}^{I}\left(\frac{\alpha_i}{\omega_i}\right)^{\alpha_i} & , \sigma = 1 \end{cases} \tag{2}$$

which can be seen as a firm-specific productivity term reflecting the benefit of access to cheaper inputs. It is also the inverse of the firm-specific ideal cost index, in the sense that cost-minimizing total cost may be expressed as $C(Q) = Q^{\frac{1}{\nu}}A^{-\frac{1}{\nu}}\Phi^{-1}$, as will be shown below.

1.1 Cost minimization

Let ω_i denote the firm-specific price for each factor of production, which the firm treats as exogenous. For a given level of output $\bar{Q}$, the firm's cost minimization problem can be written as

$$\min_{\{X_i \geq 0\}} \sum_{i=1}^{I}\omega_i X_i + \lambda\left[\bar{Q} - F(\mathbf{X})\right] \tag{3}$$

The first order conditions for cost minimization imply that $F_i(\mathbf{X}) = \lambda\omega_i\ \forall i$. Taking the ratio of first order conditions for inputs m and k, we see that in an optimum, the relative factor proportions must satisfy $\frac{\omega_k}{\omega_m} = \frac{\alpha_k}{\alpha_m}\left(\frac{X_m}{X_k}\right)^{1/\sigma}$ for all $\sigma > 0$, or re-writing,

$$X_k = X_m\left(\frac{\omega_m}{\alpha_m}\right)^{\sigma}\left(\frac{\alpha_k}{\omega_k}\right)^{\sigma},\ \sigma > 0 \tag{4}$$

It follows that cost-minimizing factor shares of total cost are constant for all levels of output and TFP levels,

$$\frac{\omega_i X_i^*}{\sum_{i=1}^{I}\omega_i X_i^*} = \frac{\omega_i\left[X_m\left(\frac{\omega_m}{\alpha_m}\right)^{\sigma}\left(\frac{\alpha_i}{\omega_i}\right)^{\sigma}\right]}{\sum_{i=1}^{I}\omega_i\left[X_m\left(\frac{\omega_m}{\alpha_m}\right)^{\sigma}\left(\frac{\alpha_i}{\omega_i}\right)^{\sigma}\right]} = \frac{\alpha_i^{\sigma}\omega_i^{1-\sigma}}{\sum_{i=1}^{I}\alpha_i^{\sigma}\omega_i^{1-\sigma}},\ \sigma > 0 \tag{5}$$

In order to obtain factor demands, we can substitute for X_i in the production constraint:

$$\bar{Q} \equiv \begin{cases} A\left(\sum_{i=1}^{I}\alpha_i\left[X_m\left(\frac{\omega_m}{\alpha_m}\right)^{\sigma}\left(\frac{\alpha_i}{\omega_i}\right)^{\sigma}\right]^{\frac{\sigma-1}{\sigma}}\right)^{\nu\frac{\sigma}{\sigma-1}} = AX_m^{\nu}\left(\frac{\omega_m}{\alpha_m}\right)^{\nu\sigma}\left[\sum_{i=1}^{I}\alpha_i^{\sigma}\omega_i^{1-\sigma}\right]^{\nu\frac{\sigma}{\sigma-1}} & , 0<\sigma\neq 1 \\ A\left[\prod_{i=1}^{I}X_m^{\alpha_i}\left(\frac{\omega_m}{\alpha_m}\right)^{\alpha_i}\left(\frac{\alpha_i}{\omega_i}\right)^{\alpha_i}\right]^{\nu} = AX_m^{\nu}\left(\frac{\omega_m}{\alpha_m}\right)^{\nu}\left[\prod_{i=1}^{n}\left(\frac{\alpha_i}{\omega_i}\right)^{\alpha_i}\right]^{\nu} & , \sigma=1 \end{cases} \tag{6}$$

and then solve for X_m^*, finding:

$$X_m^* = \begin{cases} \left(\frac{\bar{Q}}{A}\right)^{\frac{1}{\nu}}\left(\frac{\alpha_m}{\omega_m}\right)^{\sigma}\left[\sum_{i=1}^{I}\alpha_i^{\sigma}\omega_i^{1-\sigma}\right]^{\frac{\sigma}{1-\sigma}} & , 0<\sigma\neq 1 \\ \left(\frac{\bar{Q}}{A}\right)^{\frac{1}{\nu}}\left(\frac{\alpha_m}{\omega_m}\right)\prod_{i=1}^{I}\left(\frac{\omega_i}{\alpha_i}\right)^{\alpha_i} & , \sigma=1 \end{cases} \tag{7}$$

or

$$X_m^* = Q^{\frac{1}{\nu}}A^{-\frac{1}{\nu}}\Phi^{-\sigma}\left(\frac{\alpha_m}{\omega_m}\right)^{\sigma} \tag{8}$$

In order to obtain the minimum cost function, we can substitute the optimal factor demands into the cost function such that $C(\bar{Q},\boldsymbol{\omega}) = \sum_{i=1}^{n}\omega_i X_i^*$, or:

$$C(\bar{Q},\boldsymbol{\omega}) = \begin{cases} \left(\frac{\bar{Q}}{A}\right)^{\frac{1}{\nu}}\left[\sum_{i=1}^{I}\alpha_i^{\sigma}\omega_i^{1-\sigma}\right]^{\frac{1}{1-\sigma}} & , 0<\sigma\neq 1 \\ \left(\frac{\bar{Q}}{A}\right)^{\frac{1}{\nu}}\prod_{i=1}^{I}\left(\frac{\omega_i}{\alpha_i}\right)^{\alpha_i} & , \sigma=1 \end{cases} \tag{9}$$

or more simply,

$$C(\bar{Q},\boldsymbol{\omega}) = Q^{\frac{1}{\nu}}A^{-\frac{1}{\nu}}\Phi^{-1} \tag{10}$$

Then the marginal cost of output is given by

$$c(\bar{Q},\boldsymbol{\omega}) = \begin{cases} \frac{1}{\nu}A^{-\frac{1}{\nu}}\left(\sum_{i=1}^{I}\alpha_i^{\sigma}\omega_i^{1-\sigma}\right)^{\frac{1}{1-\sigma}}\bar{Q}^{\frac{1-\nu}{\nu}} & , 0<\sigma\neq 1 \\ \frac{1}{\nu}A^{-\frac{1}{\nu}}\prod_{i=1}^{I}\left(\frac{\omega_i}{\alpha_i}\right)^{\alpha_i}\bar{Q}^{\frac{1-\nu}{\nu}} & , \sigma=1 \end{cases} \tag{11}$$

or more simply,

$$c(Q,\boldsymbol{\omega}) = \frac{1}{\nu}Q^{\frac{1-\nu}{\nu}}A^{-\frac{1}{\nu}}\Phi^{-1} \tag{12}$$

Thus, it can also be seen that the ratio of average cost to marginal cost is equal to the returns to scale, ν.

Finally, recalling that marginal products are given by equation (1), we can now define cost-minimizing marginal products. It can be shown that

$$\begin{aligned} \left(\sum_{i=1}^{I}\alpha_i X_i^{\frac{\sigma-1}{\sigma}}\right)^{-1} &= A^{\frac{1}{\nu}}A^{-\frac{1}{\nu}\frac{1}{\sigma}}Q^{-\frac{1}{\nu}}Q^{\frac{1}{\nu}\frac{1}{\sigma}} \\ X_m^{-1} &= Q^{-\frac{1}{\nu}}A^{\frac{1}{\nu}}\Phi^{\sigma}\left(\frac{\omega_m}{\alpha_m}\right)^{\sigma} \\ X_m^{-\frac{1}{\sigma}} &= Q^{-\frac{1}{\nu}\frac{1}{\sigma}}A^{\frac{1}{\nu}\frac{1}{\sigma}}\Phi\left(\frac{\omega_m}{\alpha_m}\right) \\ X_m^{-\frac{1}{\sigma}}\left(\sum_{i=1}^{I}\alpha_i X_i^{\frac{\sigma-1}{\sigma}}\right)^{-1} &= A^{\frac{1}{\nu}}Q^{-\frac{1}{\nu}}\Phi\left(\frac{\omega_m}{\alpha_m}\right) \end{aligned}$$

and thus, cost-minimizing marginal products are given by

$$F_m = \nu\omega_m A^{\frac{1}{\nu}} Q^{\frac{\nu-1}{\nu}} \Phi, \quad \forall m \tag{13}$$

while the cost-minimizing output elasticity is given by

$$\frac{\partial Q}{\partial X_m}\frac{X_m^*}{Q} = \nu\omega_m A^{\frac{1}{\nu}} Q^{-\frac{1}{\nu}} \Phi \left[Q^{\frac{1}{\nu}} A^{-\frac{1}{\nu}} \Phi^{-\sigma} \left(\frac{\alpha_m}{\omega_m}\right)^{\sigma} \right] = \nu\alpha_m^{\sigma}\omega_m^{1-\sigma}\Phi^{1-\sigma} \tag{14}$$

1.2 Profit maximization

Under price-taking behavior, firms take prices as given, i.e., $P(Q) = \bar{P}$. Alternatively, we may assume that firms face downward sloping demand curves. In particular, assume that demand is isoelastic with $Q(P) = \theta^{\epsilon}P^{-\epsilon}$ denoting the demand function, and $P(Q) = \theta Q^{-1/\epsilon}$ the inverse demand function, with $\epsilon > 1$. Notice we can treat the firm as choosing Q to maximize profits, with input demands then determined based on cost minimization. Thus, we write the firm's maximization problem as

$$\max_{Q \geq 0} P(Q)\,Q - C(Q) \tag{15}$$

Let $\varepsilon(Q) = -(\partial Q/\partial P)(P/Q)$, and let $\mu(Q) = \varepsilon(Q)/(\varepsilon(Q) - 1)$. The first order condition requires $P(Q) + (\partial P/\partial Q)Q \equiv c(Q)$, or $P(Q)\,[1 - 1/\varepsilon(Q)] \equiv c(Q)$, or $P(Q) \equiv \mu(Q)c(Q)$.

In the competitive case, $\partial P/\partial Q = 0$ and we have that firms choose Q such that price equals marginal cost. In the monopolistic case, given isoelastic demand, $\varepsilon(Q) = \epsilon$ and $\mu(Q) = \mu = \epsilon/(\epsilon - 1)$. This implies the familiar constant markup over marginal cost condition and implicitly defines Q^* as that value such that $P(Q) \equiv \mu c(Q)$.

Price-taking. Cost-minimizing marginal costs are given by equation (11). Assuming $\nu < 1$, setting marginal cost equal to price and solving for quantity, optimal output levels are given by

$$Q^* = \begin{cases} A^{\eta}\,(\nu P)^{\nu\eta} \left[\left(\sum_{i=1}^{I} \alpha_i^{\sigma}\omega_i^{1-\sigma} \right)^{\frac{1}{\sigma-1}} \right]^{\nu\eta} & ,\ 0 < \sigma \neq 1 \\ A^{\eta}\,(\nu P)^{\nu\eta} \left[\prod_{i=1}^{I} \left(\frac{\alpha_i}{\omega_i}\right)^{\alpha_i} \right]^{\nu\eta} & ,\ \sigma = 1 \end{cases} \tag{16}$$

or, for all $\sigma > 0$,

$$Q^* = A^{\eta}\nu^{\nu\eta}P^{\nu\eta}\Phi^{\nu\eta} \tag{17}$$

with $\eta = \frac{1}{1-\nu}$. Notice that $\nu < 1$ implies $\eta > 1$, with $\eta \to \infty$ as $\nu \to 1$. As will be shown below, η can be seen as the inverse of the share of variable profits in revenue. Thus, as $\nu \to 1$ and $\eta \to \infty$, the variable profit share goes to 0, highlighting that decreasing returns to scale are a necessary condition for positive profits under price-taking.

Also notice that $\eta - 1 = \frac{1}{1-\nu} - \frac{1-\nu}{1-\nu} = \frac{\nu}{1-\nu} = \nu\eta$, such that $\nu\eta + 1 = \eta$. Using this fact, we see that optimal revenue, PQ^*, is given by

$$Y^* = \begin{cases} (AP)^{\eta}\nu^{\eta-1} \left[\left(\sum_{i=1}^{I} \alpha_i^{\sigma}\omega_i^{1-\sigma} \right)^{\frac{1}{\sigma-1}} \right]^{\eta-1} & ,\ 0 < \sigma \neq 1 \\ (AP)^{\eta}\nu^{\eta-1} \left[\prod_{i=1}^{I} \left(\frac{\alpha_i}{\omega_i}\right)^{\alpha_i} \right]^{\eta-1} & ,\ \sigma = 1 \end{cases} \tag{18}$$

or, for all $\sigma > 0$,

$$Y^* = A^{\eta}P^{\eta}\nu^{\eta-1}\Phi^{\eta-1} \tag{19}$$

Input demands may be calculated substituting target output levels from equation (16) into the equation for cost-minimizing factor demands in equation (7):

$$X_m^* = \begin{cases} A^{-\frac{1}{\nu}} \left(\frac{\alpha_m}{\omega_m}\right)^{\sigma} \left[\sum_{i=1}^{I} \alpha_i^{\sigma} \omega_i^{1-\sigma}\right]^{\frac{\sigma}{1-\sigma}} \left(A^{\eta} \left(\nu P\right)^{\nu\eta} \left[\left(\sum_{i=1}^{I} \alpha_i^{\sigma} \omega_i^{1-\sigma}\right)^{\frac{1}{\sigma-1}}\right]^{\nu\eta}\right)^{\frac{1}{\nu}} & , 0 < \sigma \neq 1 \\ A^{-\frac{1}{\nu}} \left(\frac{\alpha_m}{\omega_m}\right) \left[\prod_{i=1}^{I} \left(\frac{\omega_i}{\alpha_i}\right)^{\alpha_i}\right] \left(A^{\eta} \left(\nu P\right)^{\nu\eta} \left[\prod_{i=1}^{I} \left(\frac{\alpha_i}{\omega_i}\right)^{\alpha_i}\right]^{\nu\eta}\right)^{\frac{1}{\nu}} & , \sigma = 1 \end{cases}$$

which can be simplified

$$X_m^* = \begin{cases} \left(\frac{\alpha_m}{\omega_m}\right)^{\sigma} A^{\frac{\eta-1}{\nu}} \left(\nu P\right)^{\eta} \left[\left(\sum_{i=1}^{I} \alpha_i^{\sigma} \omega_i^{1-\sigma}\right)^{\frac{1}{\sigma-1}}\right]^{-\sigma} \left[\left(\sum_{i=1}^{I} \alpha_i^{\sigma} \omega_i^{1-\sigma}\right)^{\frac{1}{\sigma-1}}\right]^{\eta} & , 0 < \sigma \neq 1 \\ \left(\frac{\alpha_m}{\omega_m}\right) A^{\frac{\eta-1}{\nu}} \left(\nu P\right)^{\eta} \left[\prod_{i=1}^{I} \left(\frac{\alpha_i}{\omega_i}\right)^{\alpha_i}\right]^{-1} \left[\prod_{i=1}^{I} \left(\frac{\alpha_i}{\omega_i}\right)^{\alpha_i}\right]^{\eta} & , \sigma = 1 \end{cases}$$

Observe that $\eta - 1 = \frac{1}{1-\nu} - \frac{1-\nu}{1-\nu} = \frac{\nu}{1-\nu} = \nu\eta$, and $(\eta - 1)/\nu = \eta$. Recall that $\nu < 1$ implies $\eta > 1$, with $\eta \to \infty$ as $\nu \to 1$. Then finally,

$$X_m^* = \begin{cases} \left(\frac{\alpha_m}{\omega_m}\right)^{\sigma} \left(\nu A P\right)^{\eta} \left[\left(\sum_{i=1}^{I} \alpha_i^{\sigma} \omega_i^{1-\sigma}\right)^{\frac{1}{\sigma-1}}\right]^{\eta-\sigma} & , 0 < \sigma \neq 1 \\ \left(\frac{\alpha_m}{\omega_m}\right) \left(\nu A P\right)^{\eta} \left[\prod_{i=1}^{I} \left(\frac{\alpha_i}{\omega_i}\right)^{\alpha_i}\right]^{\eta-1} & , \sigma = 1 \end{cases} \tag{20}$$

or, for all $\sigma > 0$,

$$X_m^* = \left(\frac{\alpha_m}{\omega_m}\right)^{\sigma} \left(\nu A P\right)^{\eta} \Phi^{\eta-\sigma} \tag{21}$$

And substituting optimal output levels into the expression for cost-minimizing total costs, equation (9), we have

$$C(Q^*) \quad = \quad A^{-\frac{1}{\nu}} \left[A^{\eta} P^{\eta-1} \nu^{\eta-1} \Phi^{\nu\eta}\right]^{\frac{1}{\nu}} \Phi^{-1} = A^{\frac{\eta-1}{\nu}} P^{\frac{\eta-1}{\nu}} \nu^{\frac{\eta-1}{\nu}} \Phi^{\eta-1} = A^{\frac{\nu\eta}{\nu}} P^{\frac{\nu\eta}{\nu}} \nu^{\frac{\nu\eta}{\nu}} \Phi^{\eta-1}$$

or,

$$C(Q^*) \quad = \quad A^{\eta} P^{\eta} \nu^{\eta} \Phi^{\eta-1} \tag{22}$$

and observe that total revenue is equal to total cost times $1/\nu$, i.e., $Y^* = (1/\nu)C^*$.

Then variable profits are given by

$$\Pi_{\text{var}}^* = A^{\eta} P^{\eta} \nu^{\eta-1} \Phi^{\eta-1} - A^{\eta} P^{\eta} \nu^{\eta} \Phi^{\eta-1} = A^{\eta} P^{\eta} \nu^{\eta} \Phi^{\eta-1} \left(1 - \nu\right) \tag{23}$$

which highlights that variable profits can only be positive for $\nu < 1$. Variable profits can also be written

$$\Pi_{\text{var}}^* = (1 - \nu) Y^* \tag{24}$$

Collecting equations in their simplest forms, we have

$$Q^* = A^{\eta} P^{\eta-1} \nu^{\eta-1} \Phi^{\nu\eta} \tag{17 revisited}$$

$$Y^* = A^{\eta} P^{\eta} \nu^{\eta-1} \Phi^{\eta-1} \tag{19 revisited}$$

$$X_m^* = \left(\frac{\alpha_m}{\omega_m}\right)^{\sigma} \nu^{\eta} A^{\eta} P^{\eta} \Phi^{\eta-\sigma} \tag{21 revisited}$$

and

$$C^* = A^\eta P^\eta \nu^\eta \Phi^{\eta-1} \tag{22 revisited}$$

$$\Pi^*_{\text{var}} = A^\eta P^\eta \nu^{\eta-1} \Phi^{\eta-1}(1-\nu) \tag{23 revisited}$$

Monopolistic. In the monopolistic case, firms choose quantities such that the implied price is equal to marginal cost times a markup. Given isoelastic demand with $Q(P) = \theta^\epsilon P^{-\epsilon}$, and $P(Q) = \theta Q^{-1/\epsilon}$, optimal output levels are implied by

$$Q \quad s.t. \quad \theta Q^{-1/\epsilon} = \begin{cases} \frac{\mu}{\nu} A^{-\frac{1}{\nu}} \left(\sum_{i=1}^I \alpha_i^\sigma \omega_i^{1-\sigma}\right)^{\frac{1}{1-\sigma}} \bar{Q}^{\frac{1-\nu}{\nu}} & , 0 < \sigma \neq 1 \\ \frac{\mu}{\nu} A^{-\frac{1}{\nu}} \prod_{i=1}^I \left(\frac{\omega_i}{\alpha_i}\right)^{\alpha_i} \bar{Q}^{\frac{1-\nu}{\nu}} & \sigma = 1 \end{cases}$$

with $\mu = \epsilon/(\epsilon - 1)$, which implies that

$$Q^* = \begin{cases} A^\eta \theta^{\nu\eta} \left(\frac{\nu}{\mu}\right)^{\nu\eta} \left[\left(\sum_{i=1}^I \alpha_i^\sigma \omega_i^{1-\sigma}\right)^{\frac{1}{\sigma-1}}\right]^{\nu\eta} & , 0 < \sigma \neq 1 \\ A^\eta \theta^{\nu\eta} \left(\frac{\nu}{\mu}\right)^{\nu\eta} \left[\prod_{i=1}^I \left(\frac{\alpha_i}{\omega_i}\right)^{\alpha_i}\right]^{\nu\eta} & , \sigma = 1 \end{cases} \tag{25}$$

or, for all $\sigma > 0$,

$$Q^* = A^\eta \theta^{\nu\eta} \left(\frac{\nu}{\mu}\right)^{\nu\eta} \Phi^{\nu\eta} \tag{26}$$

with $\eta = \frac{\epsilon}{\nu+\epsilon-\epsilon\nu} = \frac{\mu}{\mu-\nu}$. In general, the sign of η is ambiguous. But as will be shown below, η can once again be seen as the inverse of the share of variable profits in revenue. Thus, positive profits requires $\eta > 0$, or equivalently, $\mu > \nu$. Notice that $\eta > 1$, with $\nu \to \mu \implies \eta \to \infty$.

In order to obtain optimal revenue, first observe that revenue is equal to $Q \times P(Q)$, so that the revenue function is $Y(Q) = \theta Q^{\frac{\epsilon-1}{\epsilon}}$. Next, observe that $\eta - 1 = \frac{\mu}{\mu-\nu} - 1 = \frac{\nu}{\mu-\nu} = \frac{\nu}{\mu}\eta$. Thus, $\theta\theta^{\frac{\nu}{\mu}\eta}$ equals $\theta\theta^{\eta-1}$ equals θ^η. Now substituting Q^* directly into the revenue function, we have

$$Y^* = \begin{cases} \theta^\eta (A^{\frac{1}{\mu}})^\eta \left(\frac{\nu}{\mu}\right)^{\eta-1} \left[\left(\sum_{i=1}^I \alpha_i^\sigma \omega_i^{1-\sigma}\right)^{\frac{1}{\sigma-1}}\right]^{\eta-1} & , 0 < \sigma \neq 1 \\ \theta^\eta (A^{\frac{1}{\mu}})^\eta \left(\frac{\nu}{\mu}\right)^{\eta-1} \left[\prod_{i=1}^I \left(\frac{\alpha_i}{\omega_i}\right)^{\alpha_i}\right]^{\eta-1} & , \sigma = 1 \end{cases} \tag{27}$$

or, for all $\sigma > 0$,

$$Y^* = \theta^\eta (A^{\frac{1}{\mu}})^\eta \left(\frac{\nu}{\mu}\right)^{\eta-1} \Phi^{\eta-1} \tag{28}$$

Moreover, it will be useful to observe that given ϵ (and θ), we can simply invert the revenue function to infer quantities:

$$Q^* = (Y^*)^{\frac{\epsilon}{\epsilon-1}} \theta^{-\frac{\epsilon}{\epsilon-1}} \tag{29}$$

In order to derive a convenient expression for optimal pricing, observe that $\frac{\nu}{\epsilon}\eta - 1 = \frac{\nu}{\epsilon+\nu-\epsilon\nu} - 1 = (\nu-1)\eta$. Then substituting Q^* into the inverse demand function, it can be shown that

$$P^* = \begin{cases} A^{-\frac{1}{\epsilon}\eta} \theta^{\eta-\nu\eta} \left(\frac{\nu}{\mu}\right)^{-\frac{\nu}{\epsilon}\eta} \left[\left(\sum_{i=1}^I \alpha_i^\sigma \omega_i^{1-\sigma}\right)^{\frac{1}{\sigma-1}}\right]^{-\frac{\nu}{\epsilon}\eta} & , 0 < \sigma \neq 1 \\ A^{-\frac{1}{\epsilon}\eta} \theta^{\eta-\nu\eta} \left(\frac{\nu}{\mu}\right)^{-\frac{\nu}{\epsilon}\eta} \left[\prod_{i=1}^I \left(\frac{\alpha_i}{\omega_i}\right)^{\alpha_i}\right]^{-\frac{\nu}{\epsilon}\eta} & , \sigma = 1 \end{cases} \tag{30}$$

or, for all $\sigma > 0$,

$$P^* = A^{-\frac{1}{\epsilon}\eta}\theta^{\eta-\nu\eta}\left(\frac{\nu}{\mu}\right)^{-\frac{\nu}{\epsilon}\eta}\Phi^{-\frac{\nu}{\epsilon}\eta} \tag{31}$$

Again, it will be useful to observe that given ϵ (and θ), we can use the demand function to infer prices from revenue:

$$P^* = \theta\left[(Y^*)^{\frac{\epsilon}{\epsilon-1}}\theta^{-\frac{\epsilon}{\epsilon-1}}\right]^{-1/\epsilon} = (Y^*)^{-\frac{1}{\epsilon-1}}\theta^{\frac{\epsilon}{\epsilon-1}} \tag{32}$$

Input demands may be calculated substituting target output levels from equation (25) into the equation for cost-minimizing factor demands in equation (7):

$$X_m^* = \begin{cases} A^{-\frac{1}{\nu}}\left(\frac{\alpha_m}{\omega_m}\right)^{\sigma}\left[\sum_{i=1}^{I}\alpha_i^{\sigma}\omega_i^{1-\sigma}\right]^{\frac{\sigma}{1-\sigma}}\left(A^{\eta}\theta^{\nu\eta}\left(\frac{\nu}{\mu}\right)^{\nu\eta}\left[\left(\sum_{i=1}^{I}\alpha_i^{\sigma}\omega_i^{1-\sigma}\right)^{\frac{1}{\sigma-1}}\right]^{\nu\eta}\right)^{\frac{1}{\nu}} & , 0 < \sigma \neq 1 \\ A^{-\frac{1}{\nu}}\left(\frac{\alpha_m}{\omega_m}\right)\left[\prod_{i=1}^{I}\left(\frac{\omega_i}{\alpha_i}\right)^{\alpha_i}\right]\left(A^{\eta}\theta^{\nu\eta}\left(\frac{\nu}{\mu}\right)^{\nu\eta}\left[\prod_{i=1}^{I}\left(\frac{\alpha_i}{\omega_i}\right)^{\alpha_i}\right]^{\nu\eta}\right)^{\frac{1}{\nu}} & , \sigma = 1 \end{cases}$$

which we can simplify

$$X_m^* = \begin{cases} \left(\frac{\alpha_m}{\omega_m}\right)^{\sigma}A^{\frac{\eta-1}{\nu}}\left(\frac{\nu}{\mu}\right)^{\eta}\theta^{\eta}\left[\left(\sum_{i=1}^{I}\alpha_i^{\sigma}\omega_i^{1-\sigma}\right)^{\frac{1}{\sigma-1}}\right]^{-\sigma}\left[\left(\sum_{i=1}^{I}\alpha_i^{\sigma}\omega_i^{1-\sigma}\right)^{\frac{1}{\sigma-1}}\right]^{\eta} & , 0 < \sigma \neq 1 \\ \left(\frac{\alpha_m}{\omega_m}\right)A^{\frac{\eta-1}{\nu}}\left(\frac{\nu}{\mu}\right)^{\eta}\theta^{\eta}\left[\prod_{i=1}^{I}\left(\frac{\alpha_i}{\omega_i}\right)^{\alpha_i}\right]^{-1}\left[\prod_{i=1}^{I}\left(\frac{\alpha_i}{\omega_i}\right)^{\alpha_i}\right]^{\eta} & , \sigma = 1 \end{cases}$$

Recall that $\eta - 1 = \frac{\mu}{\mu-\nu} - 1 = \frac{\nu}{\mu-\nu} = \frac{\nu}{\mu}\eta$, and $(\eta-1)/\nu = \frac{1}{\mu}\eta$. Then we can write $A^{\frac{\eta-1}{\nu}} = (A^{\frac{1}{\mu}})^{\eta}$, and

$$X_m^* = \begin{cases} \left(\frac{\alpha_m}{\omega_m}\right)^{\sigma}\theta^{\eta}(A^{\frac{1}{\mu}})^{\eta}\left(\frac{\nu}{\mu}\right)^{\eta}\left[\left(\sum_{i=1}^{I}\alpha_i^{\sigma}\omega_i^{1-\sigma}\right)^{\frac{1}{\sigma-1}}\right]^{\eta-\sigma} & , 0 < \sigma \neq 1 \\ \left(\frac{\alpha_m}{\omega_m}\right)\theta^{\eta}(A^{\frac{1}{\mu}})^{\eta}\left(\frac{\nu}{\mu}\right)^{\eta}\left[\prod_{i=1}^{I}\left(\frac{\alpha_i}{\omega_i}\right)^{\alpha_i}\right]^{\eta-1} & , \sigma = 1 \end{cases} \tag{33}$$

or, for all $\sigma > 0$,

$$X_m^* = \left(\frac{\alpha_m}{\omega_m}\right)^{\sigma}\theta^{\eta}(A^{\frac{1}{\mu}})^{\eta}\left(\frac{\nu}{\mu}\right)^{\eta}\Phi^{\eta-\sigma} \tag{34}$$

And substituting optimal output levels into the expression for cost-minimizing total costs, equation (9), we have

$$C(Q^*) \quad = \quad A^{-\frac{1}{\nu}}\left[A^{\eta}\theta^{\nu\eta}\left(\frac{\nu}{\mu}\right)^{\nu\eta}\Phi^{\nu\eta}\right]^{\frac{1}{\nu}}\Phi^{-1} = A^{\frac{\eta-1}{\nu}}\theta^{\eta}\left(\frac{\nu}{\mu}\right)^{\eta}\Phi^{\eta-1}$$

or,

$$C(Q^*) \quad = \quad (A^{\frac{1}{\mu}})^{\eta}\theta^{\eta}\left(\frac{\nu}{\mu}\right)^{\eta}\Phi^{\eta-1} \tag{35}$$

and observe the implication that total revenue is equal to total cost times μ/ν, i.e., $Y^* = (\mu/\nu)C^*$.

Thus, variable profits are given by

$$\Pi_{\text{var}}^* \quad = \quad Y^* - C^* = (A^{\frac{1}{\mu}})^{\eta}\theta^{\eta}\left(\frac{\nu}{\mu}\right)^{\eta-1}\Phi^{\eta-1} - (A^{\frac{1}{\mu}})^{\eta}\theta^{\eta}\left(\frac{\nu}{\mu}\right)^{\eta}\Phi^{\eta-1}$$

or

$$\Pi_{\text{var}}^* = (A^{\frac{1}{\mu}})^{\eta}\theta^{\eta}\left(\frac{\nu}{\mu}\right)^{\eta-1}\Phi^{\eta-1}\left(1-\frac{\nu}{\mu}\right) \tag{36}$$

which highlights that variable profits can only be positive when $\mu > \nu$. (Equivalently, observe that Gorodnichenko (2012) defines ν/μ as the returns to scale in the revenue function, which will imply a negative profit share in revenue if it exceeds unity; this is identical to the result here.) Variable profits can also be written

$$\Pi^*_{\text{var}} = \left(1 - \frac{\nu}{\mu}\right) Y^* \tag{37}$$

Collecting equations in their simplest forms, we have

$$Q^* = A^{\eta} \theta^{\nu\eta} \left(\frac{\nu}{\mu}\right)^{\nu\eta} \Phi^{\nu\eta} \tag{26 revisited}$$

$$Y^* = A^{\frac{\eta}{\mu}} \theta^{\eta} \left(\frac{\nu}{\mu}\right)^{\eta-1} \Phi^{\eta-1} \tag{28 revisited}$$

$$P^* = A^{-\frac{\eta}{\epsilon}} \theta^{\eta-\nu\eta} \left(\frac{\nu}{\mu}\right)^{-\frac{\nu}{\epsilon}\eta} \Phi^{-\frac{\nu}{\epsilon}\eta} \tag{31 revisited}$$

$$X^*_m = \left(\frac{\alpha_m}{\omega_m}\right)^{\sigma} A^{\frac{\eta}{\mu}} \theta^{\eta} \left(\frac{\nu}{\mu}\right)^{\eta} \Phi^{\eta-\sigma} \tag{34 revisited}$$

and

$$C^* = A^{\frac{\eta}{\mu}} \theta^{\eta} \left(\frac{\nu}{\mu}\right)^{\eta} \Phi^{\eta-1} \tag{35 revisited}$$

$$\Pi^*_{\text{var}} = A^{\frac{\eta}{\mu}} \theta^{\eta} \left(\frac{\nu}{\mu}\right)^{\eta-1} \Phi^{\eta-1} \left(1 - \frac{\nu}{\mu}\right) \tag{36 revisited}$$

1.3 Selected ratios

Factor intensity of revenue. Let $\Omega^*_m = \omega_i X^*_i / Y^*$ denote the factor of intensity of revenue of input m. Letting $\mu = 1$ in the price-taking case, then for both the monopolistic and price-taking cases, for all $\sigma > 0$, we can write

$$\Omega^*_m = \frac{\nu}{\mu} \alpha^{\sigma}_m \omega^{1-\sigma}_m \Phi^{1-\sigma} = \left(\frac{\partial Q}{\partial X_m} \frac{X^*_m}{Q}\right) / \mu \tag{38}$$

Notice that Ω^*_m is equal to the cost-minimizing output elasticity in equation (14), divided by the markup. That is, observing a given firm's factor share of revenue, we also know that firm's output elasticity up to a scale factor. Under perfect competition, Ω^*_m is exactly the firm's output elasticity, while in the presence of markups, this measure of output elasticities will be downward biased. More generally, if we assume or estimate a common markup across firms within an industry, we can recover firm-specific output elasticities. However, if the common markup assumption is untrue, then firms with higher than average markups will have downward-biased estimated output elasticities.

Alternatively, given an assumed or estimated common output elasticity at the industry level, we can recover firm-specific markups (e.g., De Loecker 2011). However, recall that firm-specific output elasticities (which result from allowing either firm-specific ω_i or α_i parameters, or both) are needed to rationalize variation in input mixes within the same detailed industry. If the common output elasticity assumption is untrue, then firms with higher than average output elasticities at a given level of factor intensity will have upward-biased markups.

Factor-output ratio / inverse average revenue product. It follows immediately that for both the monopolistic and price-taking cases, for all $\sigma > 0$, we can write

$$(y_m^*)^{-1} = \frac{X_m^*}{Y^*} = \frac{\nu}{\mu}\alpha_m^{\sigma}\omega_m^{-\sigma}\Phi^{1-\sigma} \tag{39}$$

Taking the derivative with respect to ω_m, we see that $\partial(X_m^*/Y^*)/\partial\omega_m < 0$. Thus, the inverse average revenue product should be decreasing with wage, or increasing with inverse wage. More intuitively, the implication is that the average revenue product, like the marginal revenue product, is increasing in the wage rate.

2 Comparative statics of productivity shocks

2.1 Proportional productivity shock, $A' = A(1-\tau_A)$

Consider a productivity shock of the form $A' = A(1-\tau_A)$. So that replacing A with A' and differentiating with respect to τ_A, we have

$$\frac{\partial Q^*/\partial \tau_A}{Q^*} = \frac{\partial Y^*/\partial \tau_A}{Y^*} = \frac{\partial X_m^*/\partial \tau_A}{X_m^*} = \frac{\partial \Pi_{\text{var}}^*/\partial \tau_A}{\Pi_{\text{var}}^*} = -\frac{\eta}{1-\tau_A} \tag{40}$$

$$\frac{\partial \Omega_m^*/\partial \tau_A}{\Omega_m^*} = 0 \tag{41}$$

with $\eta = \frac{1}{1-\nu}$. Recall that $\nu < 1$ implies $\eta > 1$, with $\eta \to \infty$ as $\nu \to 1$.

Under monopolistic competition, we have

$$\frac{\partial Q^*/\partial \tau_A}{Q^*} = -\frac{\eta}{1-\tau_A} \tag{42}$$

$$\frac{\partial P^*/\partial \tau_A}{P^*} = \frac{\eta/\epsilon}{1-\tau_A} \tag{43}$$

$$\frac{\partial Y^*/\partial \tau_A}{Y^*} = \frac{\partial X_m^*/\partial \tau_A}{X_m^*} = \frac{\partial \Pi_{\text{var}}^*/\partial \tau_A}{\Pi_{\text{var}}^*} = -\frac{\eta/\mu}{1-\tau_A} \tag{44}$$

$$\frac{\partial \Omega_m^*/\partial \tau_A}{\Omega_m^*} = 0 \tag{45}$$

with $\eta = \frac{\epsilon}{\nu+\epsilon-\epsilon\nu} = \frac{\mu}{\mu-\nu}$.

2.2 Additive productivity shock, $A' = A - t_A$

Consider a productivity shock of the form $A' = A - t_A$. Under price-taking behavior, we have that

$$\frac{\partial Q^*/\partial t_A}{Q^*} = \frac{\partial Y^*/\partial t_A}{Y^*} = \frac{\partial X_m^*/\partial t_A}{X_m^*} = \frac{\partial \Pi_{\text{var}}^*/\partial t_A}{\Pi_{\text{var}}^*} = -\frac{\eta}{A-t_A} \tag{46}$$

$$\frac{\partial \Omega_m^*/\partial t_A}{\Omega_m^*} = 0 \tag{47}$$

with $\eta = \frac{1}{1-\nu}$. Recall that $\nu < 1$ implies $\eta > 1$, with $\eta \to \infty$ as $\nu \to 1$.

Under monopolistic behavior, we have that

$$\frac{\partial Q^*/\partial t_A}{Q^*} = -\frac{\eta}{A-t_A} \tag{48}$$

$$\frac{\partial P^*/\partial t_A}{P^*} = \frac{\eta/\epsilon}{A-t_A} \tag{49}$$

and

$$\frac{\partial Y^*/\partial t_A}{Y^*} = \frac{\partial X_m^*/\partial t_A}{X_m^*} = \frac{\partial \Pi_{\text{var}}^*/\partial t_A}{\Pi_{\text{var}}^*} = -\frac{\eta/\mu}{A-t_A} \tag{50}$$

$$\frac{\partial \Omega_m^*/\partial t_A}{\Omega_m^*} = 0 \tag{51}$$

with $\eta = \frac{\epsilon}{\nu+\epsilon-\epsilon\nu} = \frac{\mu}{\mu-\nu}$.

3 Comparative statics of factor price shocks

3.1 Proportional factor price shock, $\omega_m' = (1+\tau_m)\omega_m$

Consider a proportional factor price shock, $\omega_m' = (1+\tau_m)\omega_m$. In this case, we can write

$$\Phi = \begin{cases} \left[\sum_{i=1}^{I-1} \alpha_i^\sigma \omega_i^{1-\sigma} + \alpha_m^\sigma \omega_m^{1-\sigma}(1+\tau_m)^{1-\sigma}\right]^{\frac{1}{\sigma-1}} & , \ 0 < \sigma \neq 1 \\ \left[\prod_{i=1}^{I} \left(\frac{\alpha_i}{\omega_i}\right)^{\alpha_i}\right](1+\tau_m)^{-\alpha_m} & , \ \sigma = 1 \end{cases} \tag{52}$$

with, for all $\sigma > 0$ (and letting $\mu = 1$ under price-taking), we have

$$\frac{\partial \Phi}{\partial \tau_m} = -\Phi \alpha_m^\sigma \omega_m^{1-\sigma}(1+\tau_m)^{-\sigma}\Phi^{1-\sigma} \tag{53}$$

$$\Phi^{-1}\frac{\partial \Phi}{\partial \tau_m} = -\alpha_m^\sigma \omega_m^{1-\sigma}(1+\tau_m)^{-\sigma}\Phi^{1-\sigma} = -\frac{\mu}{\nu}(1+\tau_m)^{-1}\Omega_m^* \tag{54}$$

Under price-taking behavior, noting that $\eta - 1 = \frac{1}{1-\nu} - \frac{1-\nu}{1-\nu} = \frac{\nu}{1-\nu} = \nu\eta$, we have that

$$\frac{\partial Q^*/\partial \tau_m}{Q^*} = \nu\eta\Phi^{-1}\frac{\partial \Phi}{\partial \tau_m} = -(\eta-1)\frac{1}{\nu}\frac{\Omega_m^*}{1+\tau_m} = -\eta\frac{\Omega_m^*}{1+\tau_m}$$

and

$$\frac{\partial Y^*/\partial \tau_m}{Y^*} = \frac{\partial \Pi_{\text{var}}^*/\partial \tau_m}{\Pi_{\text{var}}^*} = (\eta-1)\Phi^{-1}\frac{\partial \Phi}{\partial \tau_m} = -(\eta-1)\frac{1}{\nu}\frac{\Omega_m^*}{1+\tau_m} = -\eta\frac{\Omega_m^*}{1+\tau_m} \tag{55}$$

$$\frac{\partial X_m^*/\partial \tau_m}{X_m^*} = (\eta-\sigma)\Phi^{-1}\frac{\partial \Phi}{\partial \tau_m} - \frac{\sigma}{1+\tau_m} = -(\eta-\sigma)\frac{1}{\nu}\frac{\Omega_m^*}{1+\tau_m} - \frac{\sigma}{1+\tau_m} \tag{56}$$

$$\frac{\partial X_n^*/\partial \tau_m}{X_n^*} = (\eta-\sigma)\Phi^{-1}\frac{\partial \Phi}{\partial \tau_m} = -(\eta-\sigma)\frac{1}{\nu}\frac{\Omega_m^*}{1+\tau_m} \tag{57}$$

Under monopolistic behavior, we have that

$$\frac{\partial Q^*/\partial \tau_m}{Q^*} = \nu\eta\Phi^{-1}\frac{\partial \Phi}{\partial \tau_m} = -\mu\eta\frac{\Omega_m^*}{1+\tau_m}$$

$$\frac{\partial P^*/\partial \tau_m}{P^*} = -\frac{\nu}{\epsilon}\eta\Phi^{-1}\frac{\partial \Phi}{\partial \tau_m} = \frac{\mu\eta}{\epsilon}\frac{\Omega_m^*}{1+\tau_m}$$

and

$$\frac{\partial Y^*/\partial \tau_m}{Y^*} = \frac{\partial \Pi_{\text{var}}^*/\partial \tau_m}{\Pi_{\text{var}}^*} = (\eta-1)\Phi^{-1}\frac{\partial \Phi}{\partial \tau_m} = -(\eta-1)\frac{\mu}{\nu}\frac{\Omega_m^*}{1+\tau_m} = -\eta\frac{\Omega_m^*}{1+\tau_m} \tag{58}$$

$$\frac{\partial X_m^*/\partial \tau_m}{X_m^*} = (\eta-\sigma)\Phi^{-1}\frac{\partial \Phi}{\partial \tau_m} - \frac{\sigma}{1+\tau_m} = -(\eta-\sigma)\frac{\mu}{\nu}\frac{\Omega_m^*}{1+\tau_m} - \frac{\sigma}{1+\tau_m} \tag{59}$$

$$\frac{\partial X_n^*/\partial \tau_m}{X_n^*} = (\eta-\sigma)\Phi^{-1}\frac{\partial \Phi}{\partial \tau_m} = -(\eta-\sigma)\frac{\mu}{\nu}\frac{\Omega_m^*}{1+\tau_m} \tag{60}$$

Under both price-taking and monopolistic behavior, we have that

$$\frac{\partial \Omega_n^*/\partial \tau_m}{\Omega_n^*} = (1-\sigma)\Phi^{-1}\frac{\partial \Phi}{\partial \tau_m} \quad = \quad -(1-\sigma)\frac{\mu}{\nu}\frac{\Omega_m^*}{1+\tau_m} \tag{61}$$

$$\frac{\partial \Omega_m^*/\partial \tau_m}{\Omega_m^*} = (1-\sigma)\Phi^{-1}\frac{\partial \Phi}{\partial \tau_m} + \frac{1-\sigma}{1+\tau_m} \quad = \quad -(1-\sigma)\frac{\mu}{\nu}\frac{\Omega_m^*}{1+\tau_m} + \frac{1-\sigma}{1+\tau_m} \tag{62}$$

3.2 Additive factor price shock, $\omega_m' = \omega_m + t_m$

Consider input price shocks of the form $\omega_m' = \omega_m + t_m$. In this case, we can write

$$\Phi = \begin{cases} \left[\sum_{i=1}^{I-1} \alpha_i^\sigma \omega_i^{1-\sigma} + \alpha_m^\sigma (\omega_m + t_m)^{1-\sigma}\right]^{\frac{1}{\sigma-1}} & , \ 0 < \sigma \neq 1 \\ \left[\prod_{i=1}^{I-1} \left(\frac{\alpha_i}{\omega_i}\right)^{\alpha_i} \alpha_m^{\alpha_m}\right] (\omega_m + t_m)^{-\alpha_m} & , \ \sigma = 1 \end{cases} \tag{63}$$

with, for all $\sigma > 0$ (and letting $\mu = 1$ under price-taking), we have

$$\frac{\partial \Phi}{\partial t_m} \quad = \quad -\Phi \alpha_m^\sigma (\omega_m + t_m)^{-\sigma} \Phi^{1-\sigma} \tag{64}$$

$$\Phi^{-1}\frac{\partial \Phi}{\partial t_m} \quad = \quad -\alpha_m^\sigma (\omega_m + t_m)^{-\sigma} \Phi^{1-\sigma} = -\frac{\mu}{\nu}\frac{X_m^*}{Y^*} \tag{65}$$

Under price-taking behavior, noting that $\eta - 1 = \frac{1}{1-\nu} - \frac{1-\nu}{1-\nu} = \frac{\nu}{1-\nu} = \nu\eta$, we have that

$$\frac{\partial Q^*/\partial t_m}{Q^*} = \nu\eta\Phi^{-1}\frac{\partial \Phi}{\partial t_m} = -(\eta-1)\frac{1}{\nu}(y_m^*)^{-1} = -\eta(y_m^*)^{-1}$$

and

$$\frac{\partial Y^*/\partial t_m}{Y^*} = \frac{\partial \Pi_{\text{var}}^*/\partial t_m}{\Pi_{\text{var}}^*} = (\eta-1)\Phi^{-1}\frac{\partial \Phi}{\partial t_m} \quad = \quad -(\eta-1)\frac{1}{\nu}(y_m^*)^{-1} = -\eta(y_m^*)^{-1} \tag{66}$$

$$\frac{\partial X_m^*/\partial t_m}{X_m^*} = (\eta-\sigma)\Phi^{-1}\frac{\partial \Phi}{\partial t_m} - \frac{\sigma}{\omega_m + \tau_m} \quad = \quad -(\eta-\sigma)\frac{1}{\nu}(y_m^*)^{-1} - \sigma(\omega_m + t_m)^{-1} \tag{67}$$

$$\frac{\partial X_n^*/\partial t_m}{X_n^*} = (\eta-\sigma)\Phi^{-1}\frac{\partial \Phi}{\partial t_m} \quad = \quad -(\eta-\sigma)\frac{1}{\nu}(y_m^*)^{-1} \tag{68}$$

Under monopolistic behavior, we have that

$$\frac{\partial Q^*/\partial \tau_m}{Q^*} = \nu\eta\Phi^{-1}\frac{\partial \Phi}{\partial t_m} \quad = \quad -\mu\eta(y_m^*)^{-1} \tag{69}$$

$$\frac{\partial P^*/\partial \tau_m}{P^*} = -\frac{\nu}{\epsilon}\eta\Phi^{-1}\frac{\partial \Phi}{\partial t_m} \quad = \quad \frac{\mu\eta}{\epsilon}(y_m^*)^{-1} \tag{70}$$

and

$$\frac{\partial Y^*/\partial t_m}{Y^*} = \frac{\partial \Pi_{\text{var}}^*/\partial t_m}{\Pi_{\text{var}}^*} = (\eta-1)\Phi^{-1}\frac{\partial \Phi}{\partial t_m} \quad = \quad -(\eta-1)\frac{\mu}{\nu}(y_m^*)^{-1} = -\eta(y_m^*)^{-1} \tag{71}$$

$$\frac{\partial X_m^*/\partial t_m}{X_m^*} = (\eta-\sigma)\Phi^{-1}\frac{\partial \Phi}{\partial t_m} - \frac{\sigma}{\omega_m + \tau_m} \quad = \quad -(\eta-\sigma)\frac{\mu}{\nu}(y_m^*)^{-1} - \sigma(\omega_m + t_m)^{-1} \tag{72}$$

$$\frac{\partial X_n^*/\partial t_m}{X_n^*} = (\eta-\sigma)\Phi^{-1}\frac{\partial \Phi}{\partial t_m} \quad = \quad -(\eta-\sigma)\frac{\mu}{\nu}(y_m^*)^{-1} \tag{73}$$

Under both price-taking and monopolistic behavior, we have that

$$\frac{\partial\Omega_n^*/\partial\tau_m}{\Omega_n^*} = (1-\sigma)\Phi^{-1}\frac{\partial\Phi}{\partial\tau_m} \quad = \quad -(1-\sigma)\frac{\mu}{\nu}(y_m^*)^{-1} \tag{74}$$

$$\frac{\partial\Omega_m^*/\partial\tau_m}{\Omega_m^*} = (1-\sigma)\Phi^{-1}\frac{\partial\Phi}{\partial\tau_m} + \frac{1-\sigma}{\omega_m+\tau_m} \quad = \quad -(1-\sigma)\frac{\mu}{\nu}(y_m^*)^{-1} + (1-\sigma)(\omega_m+\tau_m)^{-1} \tag{75}$$

4 Comparative statics of demand shocks that do not imply changes in elasticities

4.1 Proportional demand shocks, $P' = (1-\tau_p)P$

In the price-taking case, consider a demand shock of the form $P' = (1-\tau_p)P$. Under price-taking behavior, we can write

$$\begin{aligned} Y^* =& A^\eta P^\eta (1-\tau_p)^\eta \nu^{\eta-1} \Phi^{\eta-1} \\ X_m^* =& \left(\frac{\alpha_m}{\omega_m}\right)^\sigma \nu^\eta A^\eta P^\eta (1-\tau_p)^\eta \Phi^{\eta-\sigma} \end{aligned}$$

Thus, we have

$$\frac{\partial Q^*/\partial \tau_p}{Q^*} = -\frac{\eta-1}{1-\tau_p} \tag{76}$$

$$\frac{\partial P/\partial \tau_p}{P} = -\frac{1}{1-\tau_p} \tag{77}$$

$$\frac{\partial Y^*/\partial \tau_p}{Y^*} = \frac{\partial X_m^*/\partial \tau_p}{X_m^*} = \frac{\partial \Pi_{\text{var}}^*/\partial \tau_p}{\Pi_{\text{var}}^*} = -\frac{\eta}{1-\tau_p} \tag{78}$$

with $\eta = \frac{1}{1-\nu}$. Recall that $\nu < 1$ implies $\eta > 1$, with $\eta \to \infty$ as $\nu \to 1$.

In the monopolistic case, consider a demand shock of the form $P'(Q) = (1-\tau_p)P(Q)$. Given the assumption of isoelastic demand, $P(Q) = \theta Q^{-1/\epsilon}$, it is straightforward to simply replace every instance of θ with $\theta(1-\tau_p)$, and differentiate with respect to τ_p. Thus, we can write

$$\begin{aligned} Q^* =& A^\eta (1-\tau_p)^{\nu\eta} \theta^{\nu\eta} \left(\frac{\nu}{\mu}\right)^{\nu\eta} \Phi^{\nu\eta} \\ Y^* =& A^{\frac{\eta}{\mu}} \theta^\eta (1-\tau_p)^\eta \left(\frac{\nu}{\mu}\right)^{\eta-1} \Phi^{\eta-1} \\ P^* =& A^{-\frac{\eta}{\epsilon}} \theta^{\eta-\nu\eta} (1-\tau_p)^{\eta-\nu\eta} \left(\frac{\nu}{\mu}\right)^{-\frac{\nu}{\epsilon}\eta} \Phi^{-\frac{\nu}{\epsilon}\eta} \\ X_m^* =& \left(\frac{\alpha_m}{\omega_m}\right)^\sigma A^{\frac{\eta}{\mu}} \theta^\eta (1-\tau_p)^\eta \left(\frac{\nu}{\mu}\right)^\eta \Phi^{\eta-\sigma} \end{aligned}$$

Under monopolistic competition, observe that $\eta - \nu\eta = \eta(1-\nu)$, so that we have

$$\frac{\partial Q^*/\partial \tau_p}{Q^*} = -\frac{\nu\eta}{1-\tau_p} \tag{79}$$

$$\frac{\partial P^*/\partial \tau_p}{P^*} = -\frac{\eta}{1-\tau_p}(1-\nu) \tag{80}$$

$$\frac{\partial Y^*/\partial \tau_p}{Y^*} = \frac{\partial X_m^*/\partial \tau_p}{X_m^*} = \frac{\partial \Pi_{\text{var}}^*/\partial \tau_p}{\Pi_{\text{var}}^*} = -\frac{\eta}{1-\tau_p} \tag{81}$$

with $\eta = \frac{\epsilon}{\nu+\epsilon-\epsilon\nu} = \frac{\mu}{\mu-\nu}$. Notice that for $\nu < 1$, $(\partial P^*/\partial \tau_p)/P^* < 0$, but the magnitude of the derivative will be smaller than the impacts on revenue and input demands. Then increasing returns to scale implies that prices will increase with a demand shock.

4.2 Additive shock to demand shifters, $\theta' = \theta - t_\theta$

In the monopolistic case, consider a demand shock of the form $\theta' = \theta - t_\theta$. Given the assumption of isoelastic demand, $P(Q) = \theta Q^{-1/\epsilon}$, it is straightforward to simply replace every instance of θ with $\theta - t_\theta$, and differentiate with respect to t_θ. Thus, we can write

$$\begin{aligned}
Q^* &= A^{\eta}(\theta - t_\theta)^{\nu\eta}\left(\frac{\nu}{\mu}\right)^{\nu\eta}\Phi^{\nu\eta} \\
Y^* &= A^{\frac{\eta}{\mu}}(\theta - t_\theta)^{\eta}\left(\frac{\nu}{\mu}\right)^{\eta-1}\Phi^{\eta-1} \\
P^* &= A^{-\frac{\eta}{\epsilon}}(\theta - t_\theta)^{(1-\nu)\eta}\left(\frac{\nu}{\mu}\right)^{-\frac{\nu}{\epsilon}\eta}\Phi^{-\frac{\nu}{\epsilon}\eta} \\
X_m^* &= \left(\frac{\alpha_m}{\omega_m}\right)^{\sigma} A^{\frac{\eta}{\mu}}(\theta - t_\theta)^{\eta}\left(\frac{\nu}{\mu}\right)^{\eta}\Phi^{\eta-\sigma}
\end{aligned}$$

Under monopolistic competition, observe that $\eta - \nu\eta = \eta(1-\nu)$, so that we have

$$\frac{\partial Q^*/\partial \tau_p}{Q^*} = -\frac{\nu\eta}{\theta - t_\theta} \tag{82}$$

$$\frac{\partial P^*/\partial \tau_p}{P^*} = -\frac{\eta}{\theta - t_\theta}(1-\nu) \tag{83}$$

$$\frac{\partial Y^*/\partial \tau_p}{Y^*} = \frac{\partial X_m^*/\partial \tau_p}{X_m^*} = \frac{\partial \Pi_{\text{var}}^*/\partial \tau_p}{\Pi_{\text{var}}^*} = -\frac{\eta}{\theta - t_\theta} \tag{84}$$

with $\eta = \frac{\epsilon}{\nu + \epsilon - \epsilon\nu} = \frac{\mu}{\mu - \nu}$. Notice that for $\nu < 1$, $(\partial P^*/\partial \tau_p)/P^* < 0$, but the magnitude of the derivative will be smaller than the impacts on revenue and input demands. Then increasing returns to scale implies that prices will increase with a demand shock.

5 Comparative statics of demand shocks that do imply changes in elasticities

5.1 Vertical demand distortions of the form $P'(Q) = P(Q) - t_p$

5.1.1 Implicit solutions.

In the price-taking case, it is straightforward to derive explicit solutions under vertical demand shocks of the form $P' = P - t_p$. One simply substitutes P' for P in any of price-taking solutions.

In the monopolistic case, consider a vertical demand shock, $P'(Q) = P(Q) - t_p$. Profits are given by $(P(Q) - t_p)\, Q - C(Q) = P(Q)Q - (C(Q) - t_p Q)$. Optimal output quantities $Q^*(t_p)$ in the monopolistic case must be defined implicitly as

$$Q \; s.t. \begin{cases} \theta Q^{-1/\epsilon} - \mu \left[\frac{1}{\nu} A^{-\frac{1}{\nu}} \left(\sum_{i=1}^{I} \alpha_i^{\sigma} \omega_i^{1-\sigma} \right)^{\frac{1}{1-\sigma}} Q^{\frac{1-\nu}{\nu}} \right] - \mu t_p = 0 & , 0 < \sigma \neq 1 \\ \theta Q^{-1/\epsilon} - \mu \left[\frac{1}{\nu} A^{-\frac{1}{\nu}} \prod_{i=1}^{I} \left(\frac{\omega_i}{\alpha_i} \right)^{\alpha_i} Q^{\frac{1-\nu}{\nu}} \right] - \mu t_p = 0 & , \sigma = 1 \end{cases} \tag{85}$$

By the implicit function theorem, for all $\sigma > 0$, it can be shown that

$$\partial Q^* / \partial t_p |_{t_p = 0} = -\mu \nu \eta \frac{Q^*}{P^*} \tag{86}$$

with

$$P^*(t_p) = \theta Q^*(t_p)^{-\frac{1}{\epsilon}} - t_p \tag{87}$$

$$Y^*(t_p) = \left[\theta Q^*(t_p)^{-\frac{1}{\epsilon}} - t_p \right] Q^*(t_p) = \theta Q^*(t_p)^{\frac{\epsilon-1}{\epsilon}} - t_p Q^*(t_p) \tag{88}$$

$$X_m^*(t_p) = \left[Q^*(t_p) \right]^{\frac{1}{\nu}} A^{-\frac{1}{\nu}} \left(\frac{\alpha_m}{\omega_m} \right)^{\sigma} \Phi^{-\sigma} \tag{89}$$

$$C^*(t_p) = \left[Q^*(t_p) \right]^{\frac{1}{\nu}} A^{-\frac{1}{\nu}} \Phi^{-1} \tag{90}$$

$$\Pi^*_{\text{var}}(t_p) = \theta Q^*(t_p)^{\frac{\epsilon-1}{\epsilon}} - t_p Q^*(t_p) - \left[Q^*(t_p) \right]^{\frac{1}{\nu}} A^{-\frac{1}{\nu}} \Phi^{-1} \tag{91}$$

5.1.2 Comparative statics.

In the price-taking case, consider a demand shock of the form $P' = P - t_p$. Then we have

$$\frac{\partial Q^* / \partial t_p}{Q^*} = -\frac{\eta - 1}{P - t_p} \tag{92}$$

$$\frac{\partial P / \partial t_p}{P} = -\frac{1}{P - t_p} \tag{93}$$

$$\frac{\partial Y^* / \partial t_p}{Y^*} = \frac{\partial X_m^* / \partial t_p}{X_m^*} = \frac{\partial \Pi^*_{\text{var}} / \partial t_p}{\Pi^*_{\text{var}}} = -\frac{\eta}{P - t_p} \tag{94}$$

with $\eta = \frac{1}{1-\nu}$. Recall that $\nu < 1$ implies $\eta > 1$, with $\eta \to \infty$ as $\nu \to 1$.

In the monopolistic case, consider a vertical demand shock, $P'(Q) = P(Q) - t_p$. Optimal output quantities $Q^*(t_p)$ in the monopolistic case must be defined implicitly as in equation (85), as are optimal revenues and input demands. Consider revenue from equation (88), taking the derivative with respect to t_p, evaluated where $t_p = 0$:

$$
\begin{aligned}
\left.\frac{\partial Y^*/\partial t_p}{Y^*}\right|_{t_p=0} &= \frac{1}{Y^*}\left(\left[\frac{\epsilon-1}{\epsilon}\theta\left(Q^*\right)^{-\frac{1}{\epsilon}}\right]\frac{\partial Q^*}{\partial t_p} - Q^*\right) \\
&= \frac{1}{Y^*}\left(\frac{1}{\mu}P(Q)\frac{\partial Q^*}{\partial t_p} - Q^*\right) = \frac{1}{Y^*}\left(\frac{1}{\mu}P(Q)\left[-\mu\nu\eta\frac{Q^*}{P^*}\right] - Q^*\right) \\
&= -\frac{Q^*}{Y^*}\left(\nu\eta+1\right)
\end{aligned}
$$

or finally, noting that $\nu\eta + 1 = \frac{\epsilon\nu}{\epsilon+\nu-\epsilon\nu} + \frac{\epsilon+\nu-\epsilon\nu}{\epsilon+\nu-\epsilon\nu} = \frac{\epsilon+\nu}{\epsilon+\nu-\epsilon\nu} = \eta + \frac{\nu}{\epsilon}\eta$,

$$
\left.\frac{\partial Y^*/\partial t_p}{Y^*}\right|_{t_p=0} = -\frac{\nu\eta+1}{P^*} = -\left(\nu\eta+1\right)\left(Y^*\right)^{\frac{1}{\epsilon-1}}\theta^{-\frac{\epsilon}{\epsilon-1}} \tag{95}
$$

Taking the derivative of input demands from equation (89), we have

$$
\left.\frac{\partial X_m^*/\partial t_p}{X_m^*}\right|_{t_p=0} = \frac{1}{\nu}(Q^*)^{-1}\left(-\mu\nu\eta\frac{Q^*}{P^*}\right) = -\frac{\mu\eta}{P^*} = -\mu\eta(Y^*)^{\frac{1}{\epsilon-1}}\theta^{-\frac{\epsilon}{\epsilon-1}} \tag{96}
$$

while the derivative of price in equation (87) is given by

$$
\left.\frac{\partial P^*/\partial t_p}{P^*}\right|_{t_p=0} = \frac{1}{P^*}\left(-\frac{1}{\epsilon}\frac{P^*}{Q^*}\frac{\partial Q}{\partial t_p} - 1\right) = \frac{\mu\nu\eta/\epsilon - 1}{P^*} = \left(\mu\nu\eta/\epsilon - 1\right)\left(Y^*\right)^{\frac{1}{\epsilon-1}}\theta^{-\frac{\epsilon}{\epsilon-1}} \tag{97}
$$

Finally, taking the derivative of variable profits in equation (91), it is immediate from the envelope theorem that

$$
\partial\Pi^*_{\text{var}}/\partial t_p\big|_{t_p=0} = -Q^* = -(Y^*)^{\frac{\epsilon}{\epsilon-1}}\theta^{-\frac{\epsilon}{\epsilon-1}}
$$

and

$$
\left.\frac{\partial\Pi^*_{\text{var}}/\partial t_p}{\Pi^*_{\text{var}}}\right|_{t_p=0} = -\frac{(Y^*)^{\frac{\epsilon}{\epsilon-1}}\theta^{-\frac{\epsilon}{\epsilon-1}}}{\left(\frac{\mu-\nu}{\mu}\right)Y^*} = -\theta^{-\frac{\epsilon}{\epsilon-1}}\left(\frac{\mu}{\mu-\nu}\right)(Y^*)^{\frac{1}{\epsilon-1}} = -\eta(P^*)^{-1} \tag{98}
$$

Collecting the derivatives in their simplest forms for the monopolistic case, we have

$$
\left.\frac{\partial Q^*/\partial t_p}{Q^*}\right|_{t_p=0} = -\frac{\mu\nu\eta}{P^*} \tag{99}
$$

$$
\left.\frac{\partial P^*/\partial t_p}{P^*}\right|_{t_p=0} = \frac{\mu\nu\eta/\epsilon - 1}{P^*} \tag{100}
$$

and

$$
\left.\frac{\partial Y^*/\partial t_p}{Y^*}\right|_{t_p=0} = -\frac{\nu\eta+1}{P^*} \tag{101}
$$

$$
\left.\frac{\partial X_m^*/\partial t_p}{X_m^*}\right|_{t_p=0} = -\frac{\mu\eta}{P^*} \tag{102}
$$

$$
\left.\frac{\partial\Pi^*_{\text{var}}/\partial t_p}{\Pi^*_{\text{var}}}\right|_{t_p=0} = -\frac{\eta}{P^*} \tag{103}
$$

5.2 Horizontal demand distortions of the form $Q'(P) = Q(P) - t_q$

5.2.1 Implicit solutions.

In the price-taking case, naturally there cannot be a horizontal demand shock. Thus, I focus only on monopolistic firms.

Consider horizontal demand distortions of the form $Q'(P) = Q(P) - t_q$, with corresponding inverse demand $P'(Q) = \theta(Q + t_q)^{-1/\epsilon}$. Optimal output quantities $Q^*(t_q)$ in the monopolistic case are defined implicitly as

$$Q\ s.t.\ \begin{cases} \theta(Q+t_q)^{-\frac{1}{\epsilon}} - \frac{1}{\epsilon}\theta(Q+t_q)^{-\frac{1+\epsilon}{\epsilon}}Q - \left[\frac{1}{\nu}A^{-\frac{1}{\nu}}\left(\sum_{i=1}^{I}\alpha_i^{\sigma}\omega_i^{1-\sigma}\right)^{\frac{1}{1-\sigma}}Q^{\frac{1-\nu}{\nu}}\right] = 0 & ,\ 0 < \sigma \neq 1 \\ \theta(Q+t_q)^{-\frac{1}{\epsilon}} - \frac{1}{\epsilon}\theta(Q+t_q)^{-\frac{1+\epsilon}{\epsilon}}Q - \left[\frac{1}{\nu}A^{-\frac{1}{\nu}}\prod_{i=1}^{I}\left(\frac{\omega_i}{\alpha_i}\right)^{\alpha_i}Q^{\frac{1-\nu}{\nu}}\right] = 0 & ,\ \sigma = 1 \end{cases} \tag{104}$$

By the implicit function theorem, for all $\sigma > 0$,

$$\partial Q^*/\partial t_q\big|_{t_q=0} = \frac{\nu\eta/\epsilon}{\epsilon - 1} \tag{105}$$

with

$$P^*(t_q) = \theta\left(Q^*(t_q) + t_q\right)^{-\frac{1}{\epsilon}} \tag{106}$$

$$Y^*(t_q) = \theta\left(Q^*(t_q) + t_q\right)^{-\frac{1}{\epsilon}}Q^*(t_q) \tag{107}$$

$$X_m^*(t_q) = \left[Q^*(t_q)\right]^{\frac{1}{\nu}}A^{-\frac{1}{\nu}}\left(\frac{\alpha_m}{\omega_m}\right)^{\sigma}\Phi^{-\sigma} \tag{108}$$

$$C^*(t_q) = \left[Q^*(t_q)\right]^{\frac{1}{\nu}}A^{-\frac{1}{\nu}}\Phi^{-1} \tag{109}$$

$$\Pi_{\text{var}}^*(t_q) = \theta\left(Q^*(t_q) + t_q\right)^{-\frac{1}{\epsilon}}Q^*(t_q) - \left[Q^*(t_q)\right]^{\frac{1}{\nu}}A^{-\frac{1}{\nu}}\Phi^{-1} \tag{110}$$

5.2.2 Comparative statics.

Consider a horizontal demand shock of the form $Q'(P) = Q(P) - t_q = \theta^{\epsilon}P^{-\epsilon} - t_q$, with and corresponding inverse demand $P'(Q) = \theta(Q + t_q)^{-1/\epsilon}$. Optimal output quantities $Q^*(t_q)$ in the monopolistic case must be defined implicitly as in equation (104), as are optimal revenues and input demands.

Consider revenue from equation (107), taking the derivative with respect to t_q:

$$\begin{aligned}
\left.\frac{\partial Y^*/\partial t_q}{Y^*}\right|_{t_q=0} &= \frac{1}{Y^*}\left[-\frac{1}{\epsilon}\frac{P^*}{Q^*}\left(\frac{\partial Q^*}{\partial t_q} + 1\right)Q^* + P^*\frac{\partial Q^*}{\partial t_q}\right] = \frac{P^*}{Y^*}\left[-\frac{1}{\epsilon}\left(\frac{\nu\eta/\epsilon}{\epsilon-1} + 1\right) + \frac{\nu\eta/\epsilon}{\epsilon-1}\right] \\
&= \frac{1}{Q^*}\left[-\frac{1}{\epsilon}\frac{\nu\eta/\epsilon}{\epsilon-1} - \frac{1}{\epsilon} + \frac{\nu\eta/\epsilon}{\epsilon-1}\right] = \frac{1}{Q^*}\left[\frac{\nu\eta/\epsilon}{\epsilon-1}\left(-\frac{1}{\epsilon} + 1\right) - \frac{1}{\epsilon}\right] \\
&= \frac{1}{Q^*}\left[\frac{\nu\eta/\epsilon}{\epsilon-1}\left(\frac{\epsilon-1}{\epsilon}\right) - \frac{1}{\epsilon}\right] = \frac{1}{Q^*}\left[\nu\eta/\epsilon\left(\frac{1}{\epsilon}\right) - \frac{1}{\epsilon}\right] \\
&= \frac{1}{Q^*}\frac{1}{\epsilon}\left(\frac{\nu\eta}{\epsilon} - 1\right) = \frac{1}{Q^*}\frac{1}{\epsilon}(\nu - 1)\eta
\end{aligned}$$

or finally,

$$\left.\frac{\partial Y^*/\partial t_q}{Y^*}\right|_{t_q=0} = -(1-\nu)\frac{\eta}{\epsilon}(Q^*)^{-1} = -(1-\nu)\frac{\eta}{\epsilon}(Y^*)^{-\frac{\epsilon}{\epsilon-1}}\theta^{\frac{\epsilon}{\epsilon-1}} \tag{111}$$

Taking the derivative of input demands from equation (108), we have

$$\left.\frac{\partial X_m^*/\partial t_q}{X_m^*}\right|_{t_q=0} = \frac{1}{X^*}\left[\frac{1}{\nu}X^*(Q^*)^{-1}\left(\frac{\nu\eta/\epsilon}{\epsilon-1}\right)\right] = \frac{\eta/\epsilon}{\epsilon-1}(Q^*)^{-1} = \frac{\eta/\epsilon}{\epsilon-1}(Y^*)^{-\frac{\epsilon}{\epsilon-1}}\theta^{\frac{\epsilon}{\epsilon-1}} \tag{112}$$

while the derivative of price in equation (106) is given by

$$\begin{aligned}
\left.\frac{\partial Y^*/\partial t_q}{Y^*}\right|_{t_q=0} - \left.\frac{\partial Q^*/\partial t_q}{Q^*}\right|_{t_q=0} &= -(1-\nu)\frac{\eta}{\epsilon}(Q^*)^{-1} - \frac{\nu}{\epsilon-1}\frac{\eta}{\epsilon}(Q^*)^{-1} \\
&= -\frac{\eta}{\epsilon}(Q^*)^{-1} + \nu\frac{\eta}{\epsilon}(Q^*)^{-1} - \frac{\nu}{\epsilon-1}\frac{\eta}{\epsilon}(Q^*)^{-1} \\
&= -\frac{\eta(\epsilon-1)}{\epsilon(\epsilon-1)}(Q^*)^{-1} + \frac{\nu\eta(\epsilon-1)}{(\epsilon-1)\epsilon}(Q^*)^{-1} - \frac{\nu}{\epsilon-1}\frac{\eta}{\epsilon}(Q^*)^{-1} \\
&= -\frac{\eta(\epsilon-1)}{\epsilon(\epsilon-1)}(Q^*)^{-1} + \frac{\epsilon\nu\eta-\nu\eta}{(\epsilon-1)\epsilon}(Q^*)^{-1} - \frac{\nu\eta}{(\epsilon-1)\epsilon}(Q^*)^{-1} \\
&= \frac{\epsilon\nu\eta-\nu\eta}{(\epsilon-1)\epsilon}(Q^*)^{-1} + \frac{\eta-\epsilon\eta}{\epsilon(\epsilon-1)}(Q^*)^{-1} - \frac{\nu\eta}{(\epsilon-1)\epsilon}(Q^*)^{-1} \\
&= (\epsilon\nu-\nu-\epsilon+1-\nu)\frac{\eta}{(\epsilon-1)\epsilon}(Q^*)^{-1} \\
&= \left(-\frac{\epsilon}{\eta}+1-\nu\right)\frac{\eta}{(\epsilon-1)\epsilon}(Q^*)^{-1} \\
&= -\frac{1}{\epsilon-1}(Q^*)^{-1} + \frac{1-\nu}{\epsilon-1}\frac{\eta}{\epsilon}(Q^*)^{-1}
\end{aligned}$$

such that

$$\left.\frac{\partial P^*/\partial t_q}{P^*}\right|_{t_q=0} = -\frac{1}{\epsilon-1}(Q^*)^{-1} + \frac{1-\nu}{\epsilon-1}\frac{\eta}{\epsilon}(Q^*)^{-1} \tag{113}$$

Notice that this derivative is unambiguously negative. When $\nu > 1$, this is obvious. When $\nu < 1$, observe that the derivative of revenue remains unambiguously negative, while the quantity derivative in equation (105) is unambiguously positive. Given that the derivative of revenue is the sum of derivative of quantity and prices, it must be that the derivative of price is negative and greater in absolute value than the quantity derivative, as well as greater in absolute value than the revenue derivative.

Finally, taking the derivative of variable profits in equation (91), it is immediate from the envelope theorem that

$$\left.\partial\Pi^*_{\text{var}}/\partial t_q\right|_{t_q=0} = -\frac{1}{\epsilon}P^* = -\frac{1}{\epsilon}(Y^*)^{-\frac{1}{\epsilon-1}}\theta^{\frac{\epsilon}{\epsilon-1}}$$

and

$$\left.\frac{\partial\Pi^*_{\text{var}}/\partial t_p}{\Pi^*_{\text{var}}}\right|_{t_p=0} = -\frac{1}{\epsilon}\frac{(Y^*)^{-\frac{1}{\epsilon-1}}\theta^{\frac{\epsilon}{\epsilon-1}}}{\left(\frac{\mu-\nu}{\mu}\right)Y^*} = -\frac{1}{\epsilon}\theta^{\frac{\epsilon}{\epsilon-1}}\left(\frac{\mu}{\mu-\nu}\right)(Y^*)^{-\frac{\epsilon}{\epsilon-1}} = -\frac{\eta}{\epsilon}(Q^*)^{-1} \tag{114}$$

Collecting the derivatives in their simplest forms for the monopolistic case, we have

$$\left.\frac{\partial Q^*/\partial t_q}{Q^*}\right|_{t_q=0} = \frac{\nu}{\epsilon-1}\frac{\eta}{\epsilon}(Q^*)^{-1} \tag{115}$$

$$\left.\frac{\partial P^*/\partial t_q}{P^*}\right|_{t_q=0} = -\frac{1}{\epsilon-1}(Q^*)^{-1} + \frac{1-\nu}{\epsilon-1}\frac{\eta}{\epsilon}(Q^*)^{-1} \tag{116}$$

and

$$\left.\frac{\partial Y^*/\partial t_q}{Y^*}\right|_{t_q=0} = -(1-\nu)\frac{\eta}{\epsilon}(Q^*)^{-1} \tag{117}$$

$$\left.\frac{\partial X_m^*/\partial t_q}{X_m^*}\right|_{t_q=0} = \frac{1}{\epsilon-1}\frac{\eta}{\epsilon}(Q^*)^{-1} \tag{118}$$

$$\left.\frac{\partial \Pi_{\text{var}}^*/\partial t_q}{\Pi_{\text{var}}^*}\right|_{t_q=0} = -\frac{\eta}{\epsilon}(Q^*)^{-1} \tag{119}$$