1 Dynamic System

Diminishing Island has a special tree specie. Its population $s$ grows according to the following relationship:

$$g(s) = 0.8s - 0.00001s^2$$

Cutting down a tree yields a constant marginal benefit of 10. The total cost of annual harvest depends on the size of its current stock ($s$) and the amount of harvest ($x$), i.e., the total cost is given by $TC = \frac{200000x}{s}$.

Note: the growth function is not appropriate for forestry economics (see detailed notes in Homework homepage). The appropriate growth function is a function of time rather than stock. Dynamic management of forestry is to determine when to cut down the entire stock. You may regard this problem as a dynamic system for fishery.

1. What is the maximum carrying capacity of the stock?

Solution: The maximum carrying capacity is achieved when $g(s) = 0$ and $\frac{dg}{ds} < 0$.

$$g(s) = 0.8s - 0.00001s^2 = 0 \rightarrow s_1 = 80,000 \quad \text{and} \quad s_2 = 0 \quad (1)$$

$$\frac{dg}{ds} = 0.8 - 0.00002s < 0 \rightarrow s > 40,000 \quad (2)$$

Thus, the maximum carrying capacity of the stock is $s = 80,000$.

Note: The stock will decline as its size goes up due to the limited space and nutrients. $g(s) = 0$ implies that the stock either achieves its minimum or maximum; and $\frac{dg}{ds} < 0$ implies that the stock goes down at the current value of $s$. Thus, we can tell that at the current value of $s$, the stock achieves its maximum rather than minimum.

2. What are the sizes of the stock and the annual harvest at the maximum sustainable yield?

Solution: Maximum sustainable yield is achieved when $\frac{dg}{ds} = 0$.

$$\frac{dg}{ds} = 0.8 - 0.00002s = 0 \rightarrow s = 40,000 \quad (3)$$

3. Discussion on equilibrium outcomes with different management agents

We are looking at the equilibrium at the steady state.

(a) The government believes that everyone should be allowed to use forest, and sells licenses for $6 per tree to anyone willing to buy. What are the sizes of stock and the annual harvest at equilibrium?

Solution: In this case, government charges a license fee ($L = 6$ per tree), and allows everyone who are willing to pay to cut down trees. This is an open access situation in which profits are ultimately driven to zero. Otherwise, more people will buy license and cut down trees. If a steady state is achieved, the following conditions must hold:

$$\text{zero profit condition:} \quad \text{revenue} - \text{cost} = 0 \quad (4)$$

$$\text{sustainability or steady state condition:} \quad x = g(s) \quad (5)$$

At steady state,

$$\text{profit} = TR - TC = 10x - (200000x/s + 6x) = 4x - 200000x/s = 0 \Rightarrow s = 50,000$$


Thus, at the equilibrium, the stock and the annual harvest are given by \( s = 50,000 \) and \( x = 15,000 \).

(b) Mr. Greedy runs Diminishing Island, and he cuts down trees and sells in the market to maximize the net profit. What are the sizes of stock and the annual harvest at equilibrium?

**Solution:** Mr. Greedy seeks to maximize his profit at the steady state equilibrium. Thus, the two equilibrium conditions are satisfied:

maximize profits: \( MR = MC \)  

steady state condition: \( x = g(s) \)

We express profits in terms of the stock, and those profits are maximized when the derivative with respect to \( s \) is set to zero.

\[
TR = px = pg(s) = 10(0.8s - 0.00001s^2) \Rightarrow MR = 8 - 0.0002s \\
TC = 200,000x/s = 200,000g(s)/s \ (\text{since } g(s) = x) \\
= 160,000 - 2s \ (\text{substituting } g(s)) \Rightarrow MC = -2 \\
MR = MC \Rightarrow 8 - 0.0002s = -2 \Rightarrow s = 30,000
\]

Thus, the annual harvest is given by \( x = g(s) = 0.8s - 0.00001s^2 = 15,000 \).

2 **Non-renewable and renewable resources**

There is non-renewable energy in Diminishing Island, and people in this island only live for two periods \( (t = 1, 2) \). The total benefit of energy consumption is given by \( TB_t = 150x_t - x_t^2/6 \) where \( x_t \) is energy consumption level, and extraction of this resource incurs cost \( TC = 50x_t \). Experts estimate that the total stock is 380 units. The interest rate is 20%.

1. What are the demand of energy consumption in each period?

**Solution:** The demand function \( x_t \) is derived from the total benefit function \( TB_t \) such that \( MB = p = \frac{dT B_t}{dx_t} \). Therefore, \( p = 100 - x_t \) for \( t = 1, 2 \).

2. Assume that the owner of Diminishing Island maximizes his present value of discounted net benefits in two periods. Set up his profit maximizing problem and the corresponding constraint.

**Solution:**

\[
\max_{x_1,x_2} \quad \underbrace{TB_1 - TC_1}_{\text{net benefit at } t=1} + \underbrace{TB_2 - TC_2}_{\text{discounted net benefit at } t=2} + \frac{1}{1 + r} \\
\text{s.t.} \quad x_1 + x_2 = 380 \quad \text{constraint} \tag{13}
\]

3. Derive the condition under which the optimal level of extraction in the first period is achieved, and interpret the optimal condition.

**Solution:** Substituting \( x_2 = 400 - x_1 \) into the objective function of Equation 13 yields the following:

\[
\max_{x_1} \quad (TB_1 - TC_1) + \frac{(TB_2 - TC_2)}{1 + r} \\
= (150x_1 - x_1^2/6 - 50x_1) + \frac{150x_2 - x_2^2/6 - 50x_2}{1 + r} \\
= (150x_1 - x_1^2/6 - 50x_1) + \frac{150(380 - x_1) - (380 - x_1)^2/6 - 50(380 - x_1)}{1 + r} \tag{15}
\]
Taking derivative with respect to $x_1$ and re-arranging the first order condition yields the following optimal condition:

$$
\frac{150 - x_1/3}{\text{marginal benefit}} = \frac{150 - (380 - x_1)/3 - 50}{1 + r} \frac{50}{\text{marginal future costs}} + \frac{50}{\text{marginal extraction cost}}
$$

Equation 16 shows that the optimal level of extraction in the first period is achieved when the marginal benefit equals the marginal cost. And the marginal costs includes both marginal extraction costs in this period and the marginal future costs. The marginal future costs are the discounted foregone marginal benefits from cutting down extraction and consumption in the second period that result from higher extraction and consumption in the first period.

4. Calculate current and future optimal extraction levels ($x_1$ and $x_2$) and energy prices ($p_1$ and $p_2$).

**Solution:** Solving Equation 16 yields $x_1 = 200$. Thus, $p_1 = 150 - x_1/3 = 83$; $x_2 = 380 - 200 = 180$ and $p_2 = 150 - x_2/3 = 90$.

5. Now assume that the resource is available as a common pool resource, and anyone can have access to it. Calculate extraction levels ($x_1$ and $x_2$) and prices ($p_1$ and $p_2$). Compare your answer with part (2.4).

**Solution:** In the common pool case, everyone has an access to the resource and no one can be prevented from using it. If one person decides not to extract one unit of the resource now, there is nothing to keep someone else from extracting that unit. The common thinking is “if I do not use this resource now, someone else will use it. Therefore, I lose money by not taking it now”. As long as the price for the resource is above the marginal extraction cost, there is profit to be made from extracting one more unit. Thus, extraction continues as long as the price is above the marginal extraction cost. As in a perfectly competitive market, the equilibrium quantity supplied is when price is just equal to the extraction cost.

If the price in the first period is equal to the marginal extraction cost, then $p_1 = 50$. At this price, energy extraction and consumption level is given by $x_1 = 300$ by substituting $p_1 = 50$ into the demand function. Therefore, there 380-300=80 units of energy left for the second period, i.e., $x_2 = 80$. And the price in the second period is given by $p_2 = 150 - x_2/3 = 123$.

Thus, in comparison to part 2.4, if it is common pool resource, the extraction and consumption increases and its price goes down in the first period.

6. (cont’ with part 2.4): Suppose engineers in Diminishing Island invent a back-up technology which generates energy from renewable sources. The technology can provide at any amount at a constant marginal cost $80 per unit. Calculate current and future extraction levels of non-renewable energy, energy prices, and the amount of energy produced by this back-up technology in the second period.

**Solution:** In part 2.4, the price in the second period is 90 when there is no back-up technology. At this price, it is worthwhile for a producer of the back-up technology to supply since its marginal cost is blow the price. Therefore, there are two cases in the competitive energy market we need to consider:

- If the producer of the back-up technology has no capacity limit, then the price will be driven down to the marginal cost of alternative technology since there is no longer a quantity constraint.
- If the producer of the back-up technology has capacity limit, it is possible that the price is possible to be greater than the marginal cost.
In this case, the back-up technology has no capacity limit, the price in the second period will be exactly 80. If the price in the second period is 80, the price in the first period must be such that net profit per unit rises at the rate of interest. That is, the condition for the optimal extraction of the resource must still hold.

\[ p_1 - c = \frac{p_2 - c}{1 + r} \Rightarrow p_1 = \frac{p_2 - c}{1 + r} + c = \frac{80 - 50}{1 + 0.2} + 50 = 75 \]  

(17)

Substituting prices in the demand function which yields the energy consumption in each period:

\[ x_1 = 3(150 - p_1) = 450 - 3 \times 75 = 225 \] and \[ x_2 = 3(150 - p_2) = 450 - 3 \times 80 = 210. \] Thus, in the first period, 225 units of non-renewable energy is extracted and consumed, which leaves 380-225=155 to be extracted in the second period. But the demand in the second period is \( x_2 = 210 \), which implies that 210-155=55 units of renewable energy produced by the back-up technology is consumed.

3 Essay Question: Renewable and Non-renewable Resource

Explain the difference between the optimal exploitation of a renewable and non-renewable resources in less than one page.

Solution Keys: Renewable resource has ability to grow, while implies that one can theoretically continue to use the resource forever. In this case, if demand of resource and marginal cost of extraction resource keep the same over years, at the steady state or sustainable situation of resource utilization, quantity consumed will be the same at each year, and consequentially, prices and marginal costs are the same over years. A constant price and marginal cost over time implies that the rent from resource is unchanged. While in the case of non-renewable resource, rents increases over time resulting from less stock of exhaustible resource. The interest rate play a key role in determining the optimal use of a re-newable resource. Intuitively, we have the following insights:

- if resource is growing at a rate lower than the interest rate, one is better off harvesting the resource now, selling it and investing profits.

- If the resource is growing at a rate greater than the interest rate, it makes more sense to hold onto the resource and sell it only in the future.

For example, if a fishman has to decide either catch one unit of fish in the current period or wait till next period. Catching fish at the current period yields profit \( p - c \). If he leaves fish and waits until the next period, he obtains profit from itself and its offsprings. That is, the total profit is given by \( \frac{(p-c) + (p-c)g(x)}{1+r} \) where \( g(x) \) is a growth function. At the optimal harvest level, this fishman should be indifferent between harvesting the fish in current period and waiting until the next period. That is, \( p - c = \frac{(p-c)(1+g(x))}{1+r} \), which will be hold if \( r = g(x) \).