Key Terms and Components of Dynamic Systems:

**Dynamic Systems:** Systems that contain *time* as a parameter; such systems "evolve" over time.

**State Variable:** A state variable describes the status, or "state of being," of one of the variables in the system.

**Initial Conditions:** Values that the state variables take on at the beginning of the time period of interest.

**Control Variable:** A control variable is a variable that is under the control of some individual or group.

**Random variables (noise variables):** Uncontrolled variables which can assume several values with certain probabilities.

**Constraints:** Equations (or inequalities) which limit the values that state variables or control variables can take on.

**Equation of Motion:** The equation of motion describes how a variable changes over time.

**Solution of a System:** The solution of a dynamic system is a set of equations, where the equations are in terms of the system parameters, including *time*, such that all of the original equations in the system are satisfied. Thus, a dynamic system may have many solutions, depending on the specific initial conditions of the resource.

**Objective Function:** An objective function is an equation that measures how well the system is attaining some goal or objective, usually expressed in terms of the *state variables*, *control variables*, and *parameters* of the system.
Example: Set-up for Natural Resource Dynamic System

State variables:
(St): Denotes the level of a stock at time t; (e.g., the quantity of water stored in the reservoir behind the dam at time t).
(Ut): Uncontrolled inputs, (e.g., rain, snow).
(Yt): Outputs; outcome of systems at time t; (e.g., crops produced)

Control variables:
(Xt): Inputs whose magnitudes we can choose in our attempt to reach our objectives. (e.g., the amt. of water used for irrigation).

Parameters:
(P): Items that can be taken as constant with respect to the problem at hand. (e.g., the production elasticity of irrigated water)

Equation of motion:
Next period water stock = This period water stock + rainfall - irrigation water:

\[ S_{t+1} = S_t + U_t - X_t \]

Objective Function:
\[ \max_{X_t} NPV = \sum_{t=0}^{T} \frac{B_t[Y_t(P)] - C_t(X_t)}{(1 + r)^t} \]

Dynamic Models of Nonrenewable Resources
**Nonrenewable resources** are resources that have a finite stock and that do not grow naturally.

**Key Issues:**
- Determining optimal resource allocation and pricing.
- Sources of market failure and policies to correct market failure.

\[ t = \text{time (the initial period: } t=0; \text{ the future period: } t=1) \]
\[ r = \text{interest rate} \]
\[ S_0 = \text{initial stock of nonrenewable resource} \]
\[ X_t = \text{control variable, the amt. of the resource consumed in period } t \]
\[ B(X_t) = \text{benefit of consuming } X_t \]

**Economics of scarcity:**
- **Scarcity:** Imposes an opportunity cost on using resources today. In a natural resource system, we refer to dynamic opportunity cost as a *user cost.*
- **User Cost:** The Present Value of foregone opportunity. (e.g., if you use a unit of a natural resource today, you forego the opportunity to use it tomorrow)
Nonrenewable Resources (cont.)

The User Cost decreases as \( r \) increases:
- The higher the interest rate, the less valuable tomorrow’s benefits and the smaller the opportunity cost of using more of the resource today.
- at \( r = \infty \), resources left for tomorrow are worth nothing and user cost = 0.
- Similarly, when there is enough of the resource to go around, so that scarcity is not an issue, the user cost = 0. The dynamic model yields the same outcome as two separate static models.

Discounting: The use of discounting is important in determining the optimal extraction rate of a nonrenewable resource, because the revenue a resource owner receives in period 1 is not worth as much as the revenue received in period 0.
- the NPV of benefits in period 1 in terms of the current period 0:

\[
\text{NPV} = \frac{1}{1 + r} B(X_1)
\]

Dynamic Efficiency: An allocation of resources is said to be dynamically efficient when it maximizes the NPV of benefits.

\[ \text{Max. } L = B(X) - C(X), \]
- \( B(X) \) is now a stream of benefits through time,

\[
B(X) = B_0 + \left( \frac{1}{1 + r} \right) B_1 + \left( \frac{1}{1 + r} \right)^2 B_2 + \ldots + \left( \frac{1}{1 + r} \right)^N B_N
\]
- \( C(X) \) is now a stream of costs through time
Dynamic Efficiency: The Two Period Case

**Objective function:**

\[
\max_{X_0, X_1} \text{NPV} = B(X_0) + \frac{1}{1 + r} B(X_1).
\]

**Equation of motion (constraint):**

\[ S_0 = X_0 + X_1. \]

Note: by assuming \( X_0 + X_1 \) exactly equals \( S_0 \) (resource stock is used up), we are implicitly assuming unsatiated demand.

the optimization problem is:

\[
\max_{X_0, X_1} \text{NPV} = B(X_0) + \frac{1}{1 + r} B(X_1)
\]

subject to: \( S_0 = X_0 + X_1. \)

The Lagrangian expression is:

\[
L = B(X_0) + \frac{1}{1 + r} B(X_1) + \lambda \cdot (S_0 - X_1 - X_0).
\]

To maximize the Lagrangian expression we find the F.O.C.'s:

\[
\frac{dL}{dX_0} = B_x(X_0) - \lambda = 0
\]

(1)

\[
\frac{dL}{dX_1} = B_x(X_1) \left( \frac{1}{1 + r} \right) - \lambda = 0
\]

(2)

\[
\frac{dL}{d\lambda} = S_0 - X_1 - X_0 = 0
\]

(3)
The system can be solved for $X_0$, $X_1$ and $\lambda$ in terms of the parameters of the system. An often useful step in this process is to set $\text{FOC (1} = \text{FOC (2)}$ and eliminate $\lambda$ to obtain:

$$B_x(X_0) = \frac{1}{1+r} B_x(X_1)$$

- then use (3) and (4) to solve for $X_0$ and $X_1$, and
- substitute $X_0$ into (1) to find $\lambda$.

We can find $P_0$ and $P_1$ by recalling that:

$$B_x(X_t) = \text{MB of X at time } t = \text{Price at time } t = P_t$$

Rearranging (4), we get:

$$(1 + r) \cdot B_x(X_0) = B_x(X_1)$$

Substituting $P_0$ for $B_x(X_0)$ and $P_1$ for $B_x(X_1)$, we find:

$$\frac{P_1 - P_0}{P_0} = r$$
Two-period Dynamic Efficiency (cont.)

Conclusions:

• when dynamic efficiency is met, the price increases at the rate of interest.

• the shadow price of S0, $\lambda$, is equal to $P_0$. the shadow value is also equal to the present value of $P_1$. In other words, $\lambda = P_0 = P_1/(1+r)$. Thus, the solution to the nonrenewable resource problem equates the NPV of benefits across all time periods in the horizon.

• If $P_0 > P_1/(1+r)$, the owner should extract more today; invest the money at $r$.

• If $P_0 < P_1/(1+r)$, the owner should leave more in the ground to extract tomorrow.

• the rate of return of holding resource stock in the ground is: $\text{IRR} > r$.

• Therefore, in equilibrium, it must be the case that $P_0 = P_1/(1+r)$.

- Produce today until $MB_0 = PV(MB_1)$

Note: The intuition for $\lambda$ is that, $\lambda = \text{the user cost of the resource}$! The solution to the dynamic problem equates the user cost of extracting the resource across all time periods.
A Numerical example:

Suppose \( B(X) = a \sqrt{X} \)
then \( B_X(X) = \frac{a}{2 \sqrt{X}} \).

noting that \( X_1 = S_0 - X_0 \) from (3),
\( X_0 \) can be found by using \( B_X(X) \) with eqn's (3) and (4) :

\[
\frac{a}{2 \sqrt{X_0}} = \frac{a}{2(1 + r) \sqrt{S_0 - X_0}} \quad \Rightarrow \quad \frac{S_0 - X_0}{X_0} = \frac{1}{(1 + r)^2} \quad \Rightarrow \quad X_0 = S_0 \frac{(1 + r)^2}{1 + (1 + r)^2} \tag{6}
\]

Substitute \( X_0 \) back into eqn (3) to find \( X_1 \):

\[
X_1 = \frac{S_0}{1 + (1 + r)^2} \tag{7}
\]

Substitute \( X_0 \) back into \( B_X(X_0) \) to find :

\[
P_0 = \frac{a}{2 \sqrt{\frac{(1 + (1 + r)^2)}{S_0(1 + r)^2}}} \tag{8}
\]

If \( S_0 \) increases, then both \( X_0 \) and \( X_1 \) increase
if \( r \) increases, then \( X_0 \) increases & \( X_1 \) decreases and \( P_0 \) decreases.

- if \( r = 0.1, S_0 = 100 \) and \( a = 10 \), then:
  \[
  X_0 = 54.75, \quad X_1 = 45.25, \quad P_0 = 0.68 \quad \text{and} \quad P_1 = 0.74
  \]
- If \( r \) increases to \( r = 0.5 \), then:
  \[
  X_0 = 69.3, \quad X_1 = 31.7, \quad P_0 = 0.6 \quad \text{and} \quad P_1
  \]
Two-Period Non-renewable Resource Model with Unsatiated Demand

- For P's: superscript = discount rate; subscripts = time period.
- For I's, M's, subscripts = discount rate.
- $r_2 > r_1; I_1 < I_2$.
- A lower discount rate implies:
  i) $P_0^1 > P_0^2$ Higher price in the initial period.
  ii) $P_1^1 < P_1^2$ Lower price in the second period.
  iii) $M_1 < M_2$ Less resource is used in the initial period.
When $S_0$ is so large that $B_x(X_0)$ and $\frac{1}{1+r} B_x(X_1)$ do not intersect at positive $P$, then:

- $X_0$ is solved for by setting $B_x(X_0) = 0$, and
- $X_1$ is solved for by setting $B_x(X_1) = 0$

This solution is identical to the solution of two individual static maximization problems, performed separately, in period 0 and period 1.

Note: There is no user cost here, because the MB curves fail to intersect. That is, there is no scarcity in a nonrenewable resource model with satiated demand.