In class midterm Exam
Thu Oct 28

Rules, regulations and advice

a) There are no restrictions on written materials; however you *may not* use a computer or any other electronic device.
b) If we want you to use a specific metric for a question, we'll state this explicitly. If nothing is specified about a metric, use the Pythagorian.
c) You don't have to answer the questions in order, but please number your answers carefully.
d) Some questions are significantly easier, and quicker to complete, than others. Get the easy/quick ones out of the way first before trying the longer/harder ones.
e) Use good judgement when it comes to deciding what to prove and what not to. For example, it would not be a good use of your time to prove that if $x \neq y$, then there exists $\epsilon > 0$ such that $d(x, y) > \epsilon$.
f) It's very hard to get partial credit for a question unless you write at least something down. So unless you are very fast, *don't obsess*. If you are pressed for time, it's almost always better to give quick-and-dirty answer to more questions than beautiful, perfect answers to fewer. Once you finish the exam, you can always go back at the end and improve on the quick-and-dirty answers. In particular, if a question has a yes/no answer, write down the answer, then come back and justify it later.
g) **When in doubt, always try to draw a picture. Not only will it help your intuition for the problem, it may also earn you partial credit.**

Problem 1 (20 points). Continuity:

Given $g : X \to Y$ and $f : Y \to Z$, define $h : X \to Z$ by, for $x \in X$, $h(x) = f(g(x))$. Prove or provide a counter-example to the following statements:

a) If f and g are continuous, then h is continuous.
b) If h is continuous then f and g are continuous.

If you prove the statement, you should prove it for an *arbitrary* metric. If you disprove it, your counter-example can work for any metric you like. Hint:

a) The following is a useful definition of continuity (equivalent to all of the others): $\xi : S \to T$ is continuous at s_0 if $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $s \in B(s_0, \delta)$ implies $\xi(s) \in B(\xi(s_0), \epsilon)$.
b) the following is a friendly function from $\mathbb{R}_+$ to $\mathbb{R}$. For some $n \in N$: $\xi(x) = \begin{cases} 0 & \text{if } x = 0 \\ x^{-n} & \text{otherwise} \end{cases}$

Problem 2 (20 points). Hemi-continuity:

Consider the correspondence $\psi(x) : \mathbb{R} \twoheadrightarrow \mathbb{R}$ defined by, for $x \in \mathbb{R}$:

$$\psi(x) = (-x^2, 0)$$

a) Sketch the graph of $\psi(\cdot)$ in a neighborhood of the origin.
b) Using an example, demonstrate that ψ is not UHC.
c) Suppose the lower inverse image of (a, b) (i.e. $\underline{\psi}^{-1}((a, b))$) is empty. Characterize (a, b).
d) Now suppose $\underline{\psi}^{-1}((a, b)) = \mathbb{R}\backslash\{0\}$. Characterize (a, b).
e) Is ψ LHC? If so, prove it. If not, provide a counterexample.

Problem 3 (20 points). Openness and Leo's Favorite Metric:

Fix $x > 0$ and $\epsilon > 0$, and let $B_{LFM}(x, \epsilon)$ denote the ϵ-ball about x w.r.t. Leo's favorite metric. Let δ denote some Euclidean metric. Identify a set of conditions on x and ϵ such that $B_{LFM}(x, \epsilon)$ is open in $\mathbb{R}$ w.r.t. the metric δ if and only if these conditions are satisfied. Recall the definition of LFM, $\rho(x, y) : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$:

$$\rho(x, y) = \begin{cases} |\frac{1}{x} - \frac{1}{y}| & \text{if } x, y > 0 \\ 0 & \text{if } x = y = 0 \\ \frac{1}{x} & \text{if } y = 0 \\ \frac{1}{y} & \text{if } x = 0 \end{cases}$$

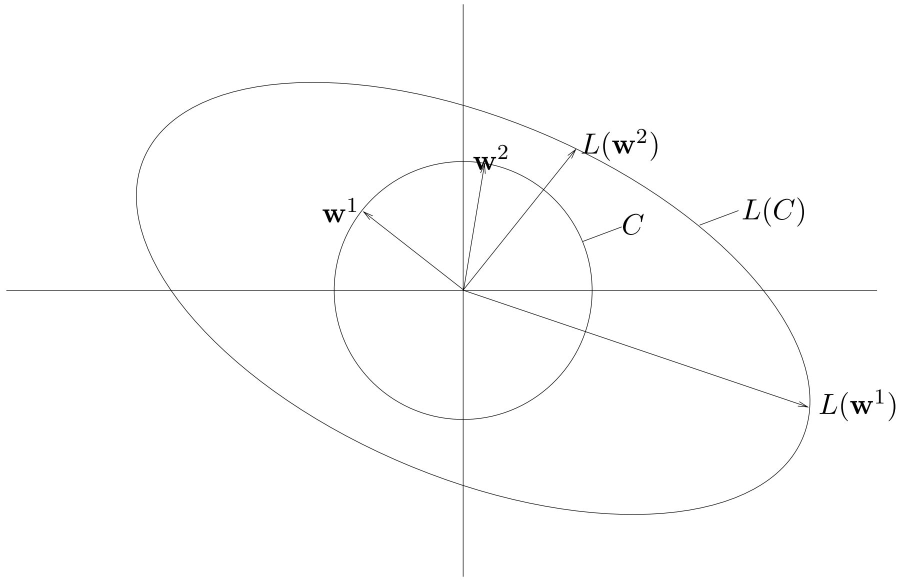

FIGURE 1

Problem 4 (20 points). Linear Algebra:

Consider the linear mapping $\mathbf{L} : \mathbb{R}^2 \to \mathbb{R}^2$ illustrated in Fig. 1. C denotes a circle with radius π. $\mathbf{L}(C)$ denotes the image of C under the mapping $\mathbf{L}$. $\mathbf{w}^1$ and $\mathbf{w}^2$ belong to C; their images under $\mathbf{L}$ are indicated in the figure. Let A denote the matrix that is uniquely associated with $\mathbf{L}$.

a) Using Fig. 2 on the back page of this exam, sketch (as accurately as you are able) two orthogonal eigen-vectors of length π for A. (Then detach the figure from the exam, **write your name on it** and hand it in with your blue book)
b) What is the maximum amount of qualitative information can you infer from the figure about the eigenvalues corresponding to these eigenvectors. (Hint: There is *exactly one* correct answer per eigen-vector to this question.) Explain your answer.
c) What is the maximum amount of qualitative information can you infer *from your answer to part b)* about the determinant of the matrix A. Explain your answer.
d) Derive your your answer to part c) using *only* information from the figure, but *without* mentioning the word "eigenvalue." Thoroughly explain your answer.

Problem 5 (20 points). Vector Spaces:

Consider the set of all convergent sequences in the metric space $(\mathbb{R}, \delta)$, where δ is a Euclidean metric.

a) Prove that this set is a vector space.
b) Write down a set of basis vectors for this space.
c) Prove that this set is a basis, i.e. that:
   i) it spans the set of convergent sequences; and
   ii) it is minimal

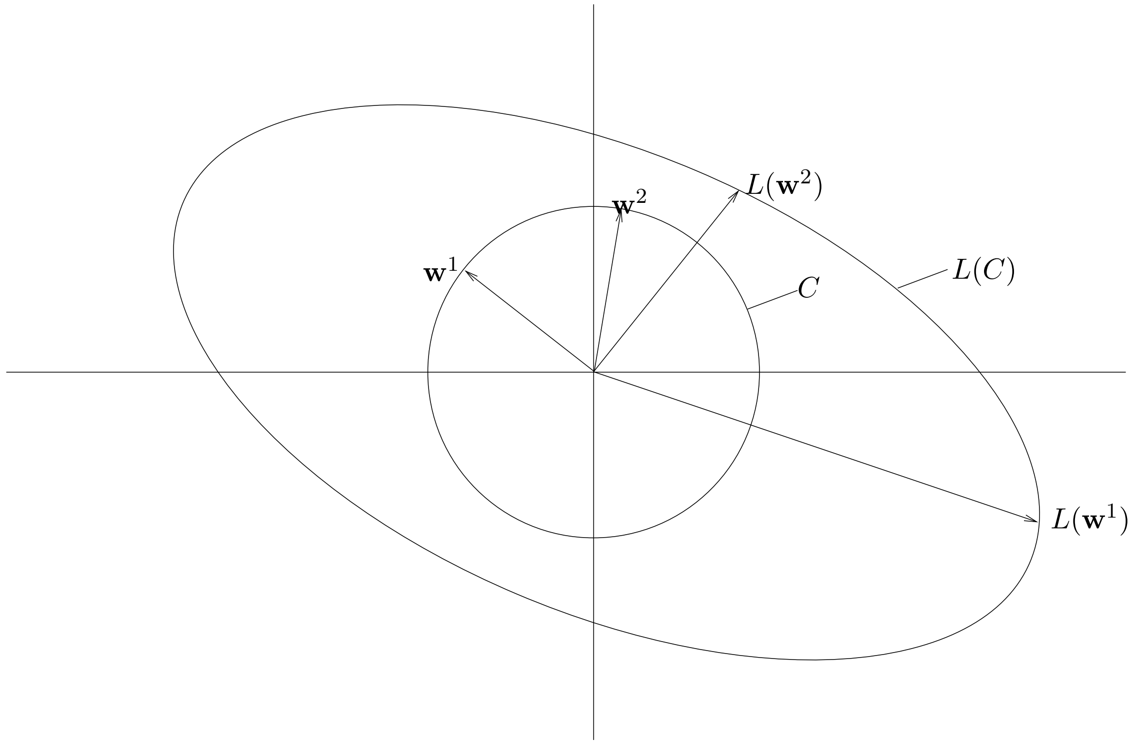

FIGURE 2. Draw your answer to Question 4a) here