IN CLASS MIDTERM EXAM
THU OCT 28

Rules, regulations and advice

a) There are no restrictions on written materials; however you *may not* use a computer or any other electronic device.
b) If we want you to use a specific metric for a question, we'll state this explicitly. If nothing is specified about a metric, use the Pythagorian.
c) You don't have to answer the questions in order, but please number your answers carefully.
d) Some questions are significantly easier, and quicker to complete, than others. Get the easy/quick ones out of the way first before trying the longer/harder ones.
e) Use good judgement when it comes to deciding what to prove and what not to. For example, it would not be a good use of your time to prove that if $x \neq y$, then there exists $\epsilon > 0$ such that $d(x, y) > \epsilon$.
f) It's very hard to get partial credit for a question unless you write at least something down. So unless you are very fast, *don't obsess*. If you are pressed for time, it's almost always better to give quick-and-dirty answer to more questions than beautiful, perfect answers to fewer. Once you finish the exam, you can always go back at the end and improve on the quick-and-dirty answers. In particular, if a question has a yes/no answer, write down the answer, then come back and justify it later.
g) **When in doubt, always try to draw a picture. Not only will it help your intuition for the problem, it may also earn you partial credit.**

Problem 1 (20 points). Metrics:

Let $\mathbf{B}$ be the set of all sequences $\mathbf{x} = (x_1, x_2, ...)$. Define $d(\mathbf{x}, \mathbf{y}) = sup\{|x_i - y_i| : i = 1, 2, ...\}$.

a) Prove that d is a metric for $\mathbf{B}$.
b) Now, let $\mathbf{F}$ be the set of all $\mathbf{x} \in \mathbf{B}$ such that $sup\{|x_i| : i = 1, 2, ...\} \leq 1$.
c) Show that $\mathbf{F}$ is closed and bounded. (Note: a set of sequences $\mathbf{F}$ is bounded if there exists $b \in \mathbb{R}$ such that for all elements of the set, the distance between that element and the zero sequence is less than b.)
d) For $n \in \mathbb{N}$, let $\mathbf{F}^n = \{\mathbf{x} \in \mathbf{B} \text{ s.t. } |x_i| \leq 1, \text{ for } 1 \leq i \leq n\}$. Provide a contra-positive proof that $F \subset \bigcup_{n \in \mathbb{N}} \mathbf{F}^n$.
e) Is $\mathbf{F}$ compact? Why or why not?

Problem 2 (20 points). Sequences:

Say whether each of the following statements is true or false. If the statement is true, provide a proof. If the statement is false, give a counter-example. Assume that $X \subset \mathbb{R}$ is endowed with the Euclidean metric.

a) Consider a sequence x_n in X.
   i) If x_n is convergent, then the sequence $y_n = x_{n+1} - x_n$ converges to 0.
   ii) If $y_n = x_{n+1} - x_n$ converges to 0, then x_n is convergent.
b) Let x_n and y_n be sequences in $\mathbf{X}$. Define the sequence z_n in $\mathbf{X}$ to be the "shuffled" sequence $\{x_1, y_1, x_2, y_2, ..., x_n, y_n, ...\}$. Then z_n is convergent if and only if x_n and y_n are both convergent with $lim\{x_n\} = lim\{y_n\}$.
c) **EXTRA CREDIT:** $y_n = \frac{x_1 + x_2 + ... + x_n}{n}$. There exists a non-convergence sequence $\{x_n\}$ for which the sequence y_n is convergent.

Problem 3 (20 points). Hemi-continuity:

a) Consider the correspondence $\xi : \mathbb{R} \to \mathbb{R}$ defined by:

$$\xi = \begin{cases} \{0\}, & x \leq 0 \\ \{-\sqrt{x}\} \cup [0, \sqrt{x}], & x > 0 \end{cases}$$

   i) Sketch a graph of the correspondence ξ.
   ii) Is ξ upper-hemi continuous, lower-hemi continuous, both, or neither? Why?
b) Next, consider the correspondence $\xi : (0, 3] \to \mathbb{R}$ defined by:

$$\psi = \begin{cases} [x^2, x), & 0 < x < 1 \\ [0, 1), & x = 1 \\ (1, x^2), & 1 < x \leq 3 \end{cases}$$

   i) Sketch a graph of the correspondence ξ.
   ii) Is ξ upper-hemi continuous, lower-hemi continuous, both, or neither? Why?

Problem 4 (20 points). Bases:

In the following, you need to justify your answer for full credit. You can get a fair bit of credit for carefully drawn pictures, providing they are carefully enough drawn. But not-graphical formalism would add value. There's a bit of duplication here, as you will see: think of it as a little gift.

a) Suppose that $S = \{\mathbf{v}^1, \mathbf{v}^2\}$ is a basis for $\mathbb{R}^2$, and consider $\mathbf{v}^3 \in \mathbb{R}^2$. Let $S_{\neg i}$ denote the set in which $\mathbf{v}^i$ is replaced by $\mathbf{v}^3$, e.g., $S_{\neg 1} = \{\mathbf{v}^2, \mathbf{v}^3\}$. Under what condition on $\mathbf{v}^3$ will
   i) *exactly one* of the $S_{\neg i}$'s be a basis for $\mathbb{R}^2$.
   ii) at least one of the $S_{\neg i}$'s be a basis set for $\mathbb{R}^2$.

b) Now suppose that $S = \{\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3\}$ is a basis for $\mathbb{R}^3$, and consider $\mathbf{v}^4 \in \mathbb{R}^3$. Again, let $S_{\neg i}$ denote the set in which $\mathbf{v}^i$ is replaced by $\mathbf{v}^4$, e.g., $S_{\neg 1} = \{\mathbf{v}^2, \mathbf{v}^3, \mathbf{v}^4\}$. Under what condition on $\mathbf{v}^4$ will
   i) *exactly one* of the $S_{\neg i}$'s be basis sets for $\mathbb{R}^3$.
   ii) *exactly two* of the $S_{\neg i}$'s be bases for $\mathbb{R}^3$.
   iii) *all three* of the $S_{\neg i}$'s be bases for $\mathbb{R}^3$.

Problem 5 (20 points). Vector Spaces:

For $n \in \{0\} \cup \mathbb{N}$, an n'th degree *polynomial* on $\mathbb{R}$ is a function of the form $f(x) = \sum_{k=0}^{n} a_k x^k$, for $k = 0, 1, ...n$, with $(a_0, a_1, ..., a_n) \in \mathbb{R}^{n+1}$ and $a_n \neq 0$. Let $\mathcal{P}(n)$ denote the set of such functions and let $\mathcal{P}$ denote the set of all polynomials on $\mathbb{R}$, i.e., $\mathcal{P} = \bigcup_n \mathcal{P}(n)$.

a) Prove that $\mathcal{P}$ an infinite-dimensional vector space. (To establish the infinite-dimensionality of the space, you just have to show that there no finite set can be a basis.)

b) Is it true that for every n, $\mathcal{P}(n)$ a vector space? If so, provide a basis for $\mathcal{P}(n)$. If it is false for some subset of $\mathbb{N}$'s, explain why not for that subset. If it is true for some subset of $\mathbb{N}$'s, prove it for that subset.

c) Consider the set $F = \bigcup_0^2 \{f \in \mathcal{P}(n), , \text{ s.t. } f(0) = f(1)\}$.
   i) Show that F is a vector space.
   ii) Prove that F has lower dimension than the space $P = \mathcal{P}(0) \cup \mathcal{P}(1) \cup \mathcal{P}(2)$.
   iii) Find a basis set for F. (Hint: Suppose $h(x) = \sum_{k=0}^{2} a_k x^k \in F$. What does this imply about $a_1 + a_2 + a_3$?)
   iv) What is the dimension of F?

Problem 6 (20 points). Rank (THIS QUESTION IS FOR EXTRA CREDIT):

Consider a pair of 2×2 matrices such that AB is the zero matrix, but BA is not.

a) Give an example of a pair A and B that satisfy the above condition. Hint: KISS (as in, keep it simply, stupid)

b) Prove that both A and B have rank 1.