In class midterm Exam
Thu Oct 28

Rules, regulations and advice

a) There are no restrictions on written materials; however you *may not* use a computer or any other electronic device.
b) If we want you to use a specific metric for a question, we'll state this explicitly. If nothing is specified about a metric, use the Pythagorian.
c) You don't have to answer the questions in order, but please number your answers carefully.
d) Some questions are significantly easier, and quicker to complete, than others. Get the easy/quick ones out of the way first before trying the longer/harder ones.
e) Use good judgement when it comes to deciding what to prove and what not to. For example, it would not be a good use of your time to prove that if $x \neq y$, then there exists $\epsilon > 0$ such that $d(x, y) > \epsilon$.
f) It's very hard to get partial credit for a question unless you write at least something down. So unless you are very fast, *don't obsess*. If you are pressed for time, it's almost always better to give quick-and-dirty answer to more questions than beautiful, perfect answers to fewer. Once you finish the exam, you can always go back at the end and improve on the quick-and-dirty answers. In particular, if a question has a yes/no answer, write down the answer, then come back and justify it later.
g) **When in doubt, always try to draw a picture. Not only will it help your intuition for the problem, it may also earn you partial credit.**

Problem 1 (30 points). Metrics:

Definition: Two metrics are *equivalent* if they define the same open sets, that is if a set is open with respect to the first metric whenever it is open with respect to the second.

a) [8 points] Given an example of two metrics on $\mathbb{R}^n$ that are not equivalent. To get full credit, you will need to *rigorously* verify your answer.

b) [12 points] Show that two metrics σ and ρ on a set X are equivalent if and only if given $\mathbf{x} \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that for all $\mathbf{y} \in X$,

$$\rho(\mathbf{x}, \mathbf{y}) < \delta \implies \sigma(\mathbf{x}, \mathbf{y}) < \epsilon \tag{1}$$

$$\sigma(\mathbf{x}, \mathbf{y}) < \delta \implies \rho(\mathbf{x}, \mathbf{y}) < \epsilon \tag{2}$$

c) [12 points] Show that the Pythagorian metric on $\mathbb{R}^n$ is equivalent to the metric ρ, defined by

$$\rho(\mathbf{x}, \mathbf{y}) = \max\{|x_i - y_i| : i = 1, ..., n\}$$

Problem 2 (40 points). Hemi-continuity:

a) Let f, g be two continuous functions mapping $S = [0,1] \subset [0,1]$ to $\mathbb{R}$ such that $f(\cdot) < g(\cdot)$. (That is, S is the universe for the domain.) Let $\xi : S \twoheadrightarrow \mathbb{R}$ be defined by $\xi(s) = \{t : f(s) < t < g(s)\}$. Identify necessary and sufficient conditions on f and g such that ξ is
   i) [15 points] upper hemi-continuous
   ii) [15 points] lower hemi-continuous (Hint: for this part, the answer key uses the neighborhood definitions of continuity, etc.)

b) [10 points] Reverse one inequality in the specification of the question in part (a), so that for the modified question, the correct answer is that "every such correspondence ξ is upper hemi-continuous." To get full credit you need to prove your answer. .

Problem 3 (30 points). Spanning:

a) Let S and T be two finite sets of vectors in $\mathbb{R}^n$.
   i) [10 points] Prove that if $S \subset T$, then $\text{span}(S) \subset \text{span}(T)$.[1]
   ii) [10 points] Prove that the $\dim(\text{span}(S \cup T) \leq \dim(\text{span}(S)) + \dim(\text{span}(T))$. Under what conditions will $\dim(\text{span}(S \cup T) = \dim(\text{span}(S)) + \dim(\text{span}(T))$?
   (For this part you may use the following result: the dimension of the span of a set of vectors S is equal to the cardinality of (i.e., number of elements in) the largest subset Q of S such that the elements of Q are linearly independent.)

b) [10 points] Let A and B be arbitrary $m \times n$ matrices. Give examples of $A \neq B$ such that
   i) $\text{rank}(A + B) < \min[\text{rank}(A), \text{rank}(B)]$
   ii) $\text{rank}(A + B) = \min[\text{rank}(A), \text{rank}(B)]$
   iii) $\text{rank}(A + B) = \text{rank}(A) + \text{rank}(B)$

[1] In the lecture notes, I referred to span(S) as "the span" of S.