In class midterm Exam - Answer key

Problem 1 (30 points). <u>Metrics</u>:

Definition: Two metrics are *equivalent* if they define the same open sets, that is if a set is open with respect to the first metric whenever it is open with respect to the second.

a) [8 points] Given an example of two metrics on $\mathbb{R}^n$ that are not equivalent. To get full credit, you will need to *rigorously* verify your answer.

Ans: The discrete metric δ on $\mathbb{R}^n$ is not equivalent to the Pythagorian metric ρ. To verify this, consider the set $S = \{0\} \subset \mathbb{R}^n$. This is open w.r.t. the discrete metric. To verify this, we need to show that 0 is an interior point of S, i.e., there exists $\epsilon > 0$ such that $B_{|\delta}(0, \epsilon) \subset S$. Let $\epsilon = 1/2$. then $B_{|\delta}(0, \epsilon) = \{0\} \subset S$. On the other hand, S is not open w.r.t. the Pythagorian metric, since 0 not an interior point of S. To verify this, observe that for all $\epsilon > 0$, $\epsilon/2 \in B_{|\rho}(0, \epsilon)$, and $\epsilon/2 \notin S$.

b) [12 points] Show that two metrics σ and ρ on a set X are equivalent if and only if given $\mathbf{x} \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that for all $\mathbf{y} \in X$,

$$\rho(\mathbf{x}, \mathbf{y}) < \delta \implies \sigma(\mathbf{x}, \mathbf{y}) < \epsilon \tag{1}$$

$$\sigma(\mathbf{x}, \mathbf{y}) < \delta \implies \rho(\mathbf{x}, \mathbf{y}) < \epsilon \tag{2}$$

Ans: We'll first show that if either (1) or (2) fails, then we can construct an open set with respect to one metric that is not open with respect to the other. Assume that (1) fails, i.e., there exists $\mathbf{x} \in X$, $\epsilon > 0$ and a sequence (y_n) such that for each n, $\rho(\mathbf{x}, y_n) < 1/n$ but $\sigma(\mathbf{x}, y_n) \geq \epsilon$. Let $U = B_\sigma(\mathbf{x}, \epsilon)$. Necessarily U is open w.r.t. σ and contains $\mathbf{x}$. However, the sequence (y_n) converges to $\mathbf{x}$ w.r.t. ρ, but none of the y_n belong to U. Hence $\mathbf{x}$ is a boundary point of U w.r.t. ρ and so cannot be open. A parallel argument can be constructed if (2) fails.

Now suppose that both (1) or (2) are satisfied. We need to show that a set U is open w.r.t. ρ iff it is open w.r.t. σ. We will do so by picking an arbitrary set U that is open w.r.t. σ and showing that an point $\mathbf{x} \in U$ is an interior point of U w.r.t. ρ. This will show that every element of U is an interior point w.r.t. ρ, and thus that U is open w.r.t. ρ. Since U is open w.r.t. σ, there exists $\epsilon > 0$ such that $B_\sigma(\mathbf{x}, \epsilon) \subset U$. From (1), there exists $\delta > 0$, such that $B_\rho(\mathbf{x}, \delta) \subset B_\sigma(\mathbf{x}, \epsilon) \subset U$. That is $\mathbf{x}$ belongs to a ρ-open subset of U, and is hence a ρ-interior point of U. A parallel argument using (1) can be constructed to show that if U is open w.r.t. ρ, then it is also open w.r.t. σ.

c) [12 points] Show that the Pythagorian metric on $\mathbb{R}^n$ is equivalent to the metric ρ, defined by

$$\rho(\mathbf{x}, \mathbf{y}) = \max\{|x_i - y_i| : i = 1, ..., n\}$$

Ans: We'll refer to the Pythag metric as σ. Fix $\mathbf{x} \in \mathbb{R}^n$ and $\epsilon > 0$.and let $\delta = \epsilon/n$. It's easier to prove that

$$\sigma(\mathbf{x}, \mathbf{y}) \geq \epsilon \implies \rho(\mathbf{x}, \mathbf{y}) \geq \delta \tag{3}$$

$$\rho(\mathbf{x}, \mathbf{y}) \geq \epsilon \implies \sigma(\mathbf{x}, \mathbf{y}) \geq \delta \tag{4}$$

Clearly, if $\rho(\mathbf{x},\mathbf{y}) \geq \epsilon$ then $\sigma(\mathbf{x},\mathbf{y}) = \sqrt{\epsilon^2 + K}$, for some nonnegative number K. Hence $\sigma(\mathbf{x},\mathbf{y}) \geq \epsilon$. On the other hand, if $\sigma(\mathbf{x},\mathbf{y}) \geq \epsilon$ then necessarily $|x_i - y_i| \geq \epsilon/\sqrt{n} > \epsilon/n$, for at least one i. But this implies that $\rho(\mathbf{x},\mathbf{y}) > \delta$.

Problem 2 (40 points). Hemi-continuity:

a) Let f, g be two continuous functions mapping $S = [0,1] \subset [0,1]$ to $\mathbb{R}$ such that $f(\cdot) < g(\cdot)$. (That is, S is the universe for the domain.) Let $\xi : S \twoheadrightarrow \mathbb{R}$ be defined by $\xi(s) = \{t : f(s) < t < g(s)\}$. Identify necessary and sufficient conditions on f and g such that ξ is

i) [15 points] upper hemi-continuous

Ans: Note that since ξ is not compact valued, the sequential definition of upper hemi continuity is not applcable. Answer based on inverse image definitions: ξ is u.h.c. iff f and g are both constant functions.

(i) **Sufficiency:** Since $f < g$, necessarily $a < b$ and $\xi(s) = (a,b)$, for all s. In this case for O open in $\mathbb{R}$, $\bar{\xi}^{-1}(0) = \begin{cases} S & \text{if } (a,b) \subset O \\ \emptyset & \text{if } (a,b) \not\subseteq O \end{cases}$. In either case $\bar{\xi}^{-1}(O)$ is open.

Necessity: Suppose that f is not constant. Since S is compact and f is continuous, it follows from Weierstrass's theorem that $\exists \bar{s} \in S$, such that $f(\bar{s}) \geq f(s)$, for all $s \in S$, and moreover, $\exists s \in S$ such that $f(s) < f(\bar{s})$. Assume w.l.o.g. that there exists $\delta > 0$ such that $f(s) < f(\bar{s})$, for all $s \in [\bar{s} - \delta, \bar{s})$. For $O = (f(\bar{s}), g(\bar{s}))$, $\bar{\xi}^{-1}(O)$ is not open. since for all $s \in [\bar{s} - \delta, \bar{s})$, $\xi(s) \not\subseteq O$.

ii) [15 points] lower hemi-continuous (Hint: for this part, the answer key uses the neighborhood definitions of continuity, etc.)

Ans: Answer based on neighborhood definitions: Every such ξ is lower-hemi-continuous. To verify this, fix $\bar{s} \in S$, an open set $U' \subset \mathbb{R}$ such that $\xi(\bar{s}) \cap U' \neq \emptyset$. Let $U = \xi(\bar{s}) \cap U'$. Since U is the intersection of two open sets, it is open. Therefore we can pick $t \in U$ and $\epsilon > 0$ such that $f(\bar{s}) + \epsilon < t < f(\bar{s}) - \epsilon$. Since g is continuous, there exists a neighborhood V^f of $\bar{s}$ such that for all $s \in V^f$, $f(s) \in B(f(\bar{s}), \epsilon)$. Similarly, since g is continuous, there exists a neighborhood V^g of $\bar{s}$ such that for all $s \in V^g$, $g(s) \in B(g(\bar{s}), \epsilon)$. Now define $V^{fg} = V^f \cap V^g$. Since V^f and V^g are open, so also is V^{fg}. For all $s \in V^{fg}$, $\big(f(\bar{s}) + \epsilon, g(\bar{s}) - \epsilon\big) \subset \xi(s)$, hence $t \in \xi(s)$. Since $t \in U$, $U \neq \emptyset$. To summarize, for an arbitrarily chosen open set U' such that $\xi(\bar{s}) \cap U' \neq \emptyset$, we have identified a neighborhood V^{fg} of $\bar{s}$ such that for all $s \in V^{fg}$, $\xi(s) \cap U' \neq \emptyset$.

Answer based on sequential definitions: Pick $\bar{s} \in S$ and $t \in \xi(\bar{s})$. Since $\xi(\bar{s}) := \big(f(\bar{s}), g(\bar{s})\big)$ is open, there exists $\epsilon > 0$ such that $t \in \big(f(\bar{s}) + \epsilon, g(\bar{s}) - \epsilon\big)$. Since f and g are continuous, there exists $\delta > 0$ such that for $s \in B(\bar{s}, \delta)$, $f(s) < f(\bar{s}) + \epsilon$ and $g(s) > g(\bar{s}) - \epsilon$. Hence for $s \in B(\bar{s}, \delta)$, $t \in \xi(s)$. Now let $\{s_n\} \to \bar{s}$ and for each n set $t^n = \begin{cases} t & \text{if } s^n \in B(\bar{s,\delta}) \\ (f(s) + g(s))/2 & \text{otherwise} \end{cases}$. By construction $t^n \in \xi(s^n)$ for all n and $\lim_n t^n = t$. Hence ξ satisfies the sequential definition of lower hemi-continuity.

b) [10 points] Reverse one inequality in the specification of the question in part (a), so that for the modified question, the correct answer is that "every such correspondence ξ is upper hemi-continuous." To get full credit you need to prove your answer.

Ans: Answer based on inverse image definition: "Let f, g be two continuous functions mapping $S = [0,1]$ to $\mathbb{R}$ such that $f(\cdot) > g(\cdot)$." Now consider the correspondence $\xi : S \twoheadrightarrow \mathbb{R}$ defined by $\xi(s) = \{t : f(s) < t < g(s)\}$. For all s, since $f(s) > g(s)$, $\xi(s) = \emptyset$. Hence for any open set $O \subset \mathbb{R}$, $\bar{\xi}^{-1}(0) = \begin{cases} S & \text{if } O = \emptyset \\ \emptyset & \text{if } O \neq \emptyset \end{cases}$. In either case $\bar{\xi}^{-1}(O)$ is open (since S is open in S.)
Answer based on inverse sequential definition: In this case, lower-hemi continuity holds vacuously. For $s \in S$, $\xi(s) = \emptyset$, so there is no $t \in \xi(s)$ to which we are required to check the sequential criterion.

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Problem 3 (30 points). Spanning:

a) Let S and T be two finite sets of vectors in $\mathbb{R}^n$.

i) [10 points] Prove that if $S \subset T$, then $\text{span}(S) \subset \text{span}(T)$.[1]

Ans: Suppose that $\mathbf{v} \in \mathbb{R}^n$ belongs to the span of $S = \{\mathbf{s}^1, ..., \mathbf{s}^{\#S}\}$. Then there exists a vector $\alpha \in \mathbb{R}^{\#S}$ such that $\mathbf{v} = \sum_{k=1}^{\#S} \alpha_k \mathbf{s}^k$. Since $S \subset T$, $\mathbf{v} = \sum_{k=1}^{\#S} \alpha_k \mathbf{s}^k + \sum_{r=1}^{\#T \setminus S} 0\mathbf{t}^r$, i.e., $\mathbf{v}$ can be written as a linear combination of the elements of T. We have thus proved that $\mathbf{v}$ belongs to the span of T.

ii) [10 points] Prove that the $\dim(\text{span}(S \cup T) \leq \dim(\text{span}(S)) + \dim(\text{span}(T))$. Under what conditions will $\dim(\text{span}(S \cup T) = \dim(\text{span}(S)) + \dim(\text{span}(T))$?
(For this part you may use the following result: the dimension of the span of a set of vectors S is equal to the cardinality of (i.e., number of elements in) the largest subset Q of S such that the elements of Q are linearly independent.)

Ans: If no member of S can be written as a linear combination of the elements of T, then $\dim(\text{span}(S \cup T) = \dim(\text{span}(S)) + \dim(\text{span}(T))$. More generally, some element of S will be expressasble as a linear combination of the elements of T, in which case $\dim(\text{span}(S \cup T) < \dim(\text{span}(S)) + \dim(\text{span}(T))$.

b) [10 points] Let A and B be arbitrary $m \times n$ matrices. Give examples of $A \neq B$ such that

i) $\text{rank}(A + B) < \min[\text{rank}(A), \text{rank}(B)]$

Ans: Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ so that $A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then $\text{rank}(A) = \text{rank}(B) = 2$, while $\text{rank}(A + B) = 0$.

ii) $\text{rank}(A + B) = \min[\text{rank}(A), \text{rank}(B)]$

[1] In the lecture notes, I referred to $\text{span}(S)$ as "the span" of S.

Ans: Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ so that $A + B = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$. Then rank$(A) =$ rank$(B) = 2$, while rank$(A + B) = 2$.

iii) $\operatorname{rank}(A + B) = \operatorname{rank}(A) + \operatorname{rank}(B)$

Ans: Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ so that $A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then rank$(A) =$ rank$(B) = 1$, while rank$(A + B) = 2$.