In class midterm Exam
Thu Oct 27 2011

Rules, regulations and advice

a) There are no restrictions on written materials; however you *may not* use a computer.
b) If we want you to use a specific metric for a question, we'll state this explicitly. If nothing is specified about a metric, use the Pythagorian.
c) You don't have to answer the questions in order, but please number your answers carefully.
d) Some questions are significantly easier, and quicker to complete, than others. Get the easy/quick ones out of the way first before trying the longer/harder ones.
e) Use good judgement when it comes to deciding what to prove and what not to. For example, it would not be a good use of your time to prove that if $x \neq y$, then there exists $\epsilon > 0$ such that $d(x, y) > \epsilon$.
f) It's very hard to get partial credit for a question unless you write at least something down. So unless you are very fast, *don't obsess*. If you are pressed for time, it's almost always better to give quick-and-dirty answer to more questions than beautiful, perfect answers to fewer. Once you finish the exam, you can always go back at the end and improve on the quick-and-dirty answers. In particular, if a question has a yes/no answer, write down the answer, then come back and justify it later.

Problem 1 (14 points). Analysis:

a) Prove by induction that $\frac{n^2+5n+1}{2} \in \mathbb{Q}$ for $n \in \mathbb{N}, n \geq 2$.
b) True or False: For a closed set A in an arbitrary metric space, $\text{cl}(\text{int}(A)) = A$. If true, prove it. If false, provide a counterexample.

Problem 2 (15 points). Hemi-continuity:
Consider the correspondence $\Psi : \mathbb{R} \to \mathbb{R}$

$$\Psi(x) = \begin{cases} \mathbb{Q} & \text{if } x > 0 \\ \mathbb{Q} \cap [0, 1) & \text{if } x \leq 0 \end{cases}$$

a) What is the lower inverse image of $V = (1, 2)$? What is the upper inverse image of $V = (1, 2)$
b) Is Ψ UHC? LHC? If so, briefly explain how you would prove it (full proof not required). If not, provide a counterexample. Hint: there's only one point in the domain that's at issue.
c) On what subset of $\mathbb{R}$ is Ψ compact valued?

Problem 3 (14 points). Maximization:
Consider the space $\mathbb{R}^n_+$ under the Euclidean metric. Let $u : \mathbb{R}^n_+ \to \mathbb{R}$ be a continuous function, and consider fixed parameters $p \in \mathbb{R}^n_{++}$, $w \in \mathbb{R}_{++}$. Are the following problems guaranteed to have solutions? If yes, explain why (a fully extensive proof is not necessary, just provide the relevant conditions and theorems that apply). If not, provide a counterexample.

a) $\max_{x \in \mathbb{R}^n_+} u(x)$ subject to $p \cdot x \leq w$.
b) $\max_{x \in \mathbb{R}^n_+} wu(x) - p \cdot x$ subject to $x \geq 0$ (i.e. all elements of x are weakly positive).

Problem 4 (14 points). Compactness:

Prove or find a counterexample: in an arbitrary metric space (X, d), the finite union of compact sets is compact.

Problem 5 (14 points). Vector spaces:

True/False. Determine whether the following sets are vector spaces. If they are, prove it. If not, demonstrate why.

a) The set of all positive definite $n \times n$ matrices.
b) For a fixed vector $v \in \mathbb{R}^n$, the set $W = \{A_{n\times n} : v \text{ is an eigenvector of } A\}$.

Problem 6 (15 points). Vector subspaces:

Consider a fixed $(a, b, c) \in \mathbb{R}^3$ and let$S_{a,b,c} = \{(x, y, z) \in \mathbb{R}^3 : ax + by + cz = 0\}$

a) Show that $S_{a,b,c}$ is a vector subspace of $\mathbb{R}^3$
b) True or false: $S_{1,1,1} \cup S_{1,2,3}$ is a vector subspace of $\mathbb{R}^3$. If true, provide a proof. If false, show by counterexample that some property of a vector subspace is violated.
c) Consider the set $A_{1,2,1} = \{(x, y, z) \in \mathbb{R} : x + 2y + z = 1\}$. True or false: $A_{1,2,1} \cap S_{1,1,1}$ a vector subspace of $\mathbb{R}^3$. If true, provide a proof. If false, show by counterexample that some property of a vector subspace is violated.

Problem 7 (14 points). Bases:

For each of the following sets of sequences in $\mathbb{R}$, can you find a basis for it? If so, what is it? If not, why not?

a) $\{\{x_n\} : x_n \neq 0 \Leftrightarrow n \text{ is even}\}$
b) $\{\{x_n\} : x_n = 1 \Rightarrow n \text{ is odd}\}$