IN CLASS MIDTERM EXAM - ANSWER KEY

Problem 1 (14 points). Analysis:

a) Prove by induction that $\frac{n^2+5n+1}{2} \in \mathbb{Q}$ for $n \in \mathbb{N}, n \geq 2$.

Ans: Base case: $n = 2$: $\frac{n^2+5n+1}{2} = 5$. Induction step: Assume $\frac{n^2+5n+1}{2} = q \in \mathbb{Q}$. Then consider the case $n+1$:

$$\begin{aligned}
\frac{(n+1)^2+5(n+1)+1}{2} &= \frac{n^2+2n+1+5n+5+1}{2} \\
&= \frac{(n^2+5n+1)+2(n+3)}{2} \\
&= \frac{n^2+5n+1}{2} + \frac{2(n+3)}{2} \\
&= q+n+3 \in \mathbb{Q}
\end{aligned}$$

b) True or False: For a closed set A in an arbitrary metric space, $\text{cl}(\text{int}(A)) = A$. If true, prove it. If false, provide a counterexample.

Ans: False. Consider $A = x \in \mathbb{R}$ in Euclidean space. Then int $A = \emptyset$ and cl $(\emptyset) = \emptyset \neq x$.

Problem 2 (15 points). Hemi-continuity:
Consider the correspondence $\Psi : \mathbb{R} \to \mathbb{R}$

$$\Psi(x) = \begin{cases} \mathbb{Q} & \text{if } x > 0 \\ \mathbb{Q} \cap [0,1) & \text{if } x \leq 0 \end{cases}$$

a) What is the lower inverse image of $V = (1,2)$? What is the upper inverse image of $V = (1,2)$

Ans: The lower inverse image of V is $\mathbb{R}_{++}$, and the upper inverse image of V is the empty set.

b) Is Ψ UHC? LHC? If so, briefly explain how you would prove it (full proof not required). If not, provide a counterexample. Hint: there's only one point in the domain that's at issue.

Ans: Ψ is not UHC. Consider the point $x = 0$ and the open interval $(-1,2)$. This interval contains $\Psi(0)$, but any open neighborhood around $x = 0$ will map into all of $\mathbb{Q}$, which is not contained in $(-1,2)$.
Ψ is LHC. To verify this, consider any nonempty open set V in $\mathbb{R}$. Note that V cannot be a single point, since this would be a closed set. Then by the density of $\mathbb{Q}$ in $\mathbb{R}$, there exists some $q \in Q$ such that $q \in V$. Either $q \in \mathbb{Q} \cap [0,1)$ or $q \in \mathbb{Q} \backslash (\mathbb{Q} \cap [0,1))$, but not both. If the former, then $q \in \Psi(x)$ for all $x \in \mathbb{R}$, so the lower inverse image of V is $\mathbb{R}$, which is open. If the latter, then $q \in \Psi(x)$ only for $x > 0$, so the lower inverse image of V is $\mathbb{R}_{++}$, which is also open. Thus, the lower inverse image of any open set V is open, and Ψ is LHC.

c) On what subset of $\mathbb{R}$ is Ψ compact valued?

Ans: None. For $x > 0$, $\Psi(x)$ is not bounded and not compact. For $x \leq 0$, $\Psi(x) = \mathbb{Q} \cap [0, 1)$. Note that 1 is an accumulation point of this set, but is not contained in the set, thus the set $\mathbb{Q} \cap [0, 1)$ is not closed and therefore not compact.

Problem 3 (14 points). Maximization:

Consider the space $\mathbb{R}^n_+$ under the Euclidean metric. Let $u : \mathbb{R}^n_+ \to \mathbb{R}$ be a continuous function, and consider fixed parameters $p \in \mathbb{R}^n_{++}$, $w \in \mathbb{R}_{++}$. Are the following problems guaranteed to have solutions? If yes, explain why (a fully extensive proof is not necessary, just provide the relevant conditions and theorems that apply). If not, provide a counterexample.

a) $\max_{x \in \mathbb{R}^n_+} u(x)$ subject to $p \cdot x \leq w$.

Ans: Yes, the problem has a solution. First, note that the constraint set, $\{x \in \mathbb{R}^n_+ : p \cdot x \leq w\}$ is a closed and bounded subset of our domain $\mathbb{R}^n$, thus it is compact. In addition, the problem states that u is continuous and maps to $\mathbb{R}$. Thus, by the Weierstrauss Theorem, u attains a global maximum on $\mathbb{R}^n$.

b) $\max_{x \in \mathbb{R}^n_+} wu(x) - p \cdot x$ subject to $x \geq 0$ (i.e. all elements of x are weakly positive).

Ans: No, this problem may not have a solution. In this case, the issue is that our domain is unbounded. Consider the function $u(x) = x^2$ on $\mathbb{R}$ with $w = 1$ and $p = 1$. Then our maximization problem becomes $\max_{x \in \mathbb{R}^n_+} x^2 - x$ which approaches infinity as x approaches infinity.

Problem 4 (14 points). Compactness:

Prove or find a counterexample: in an arbitrary metric space (X, d), the finite union of compact sets is compact.

Ans: True. Let $\mathcal{V} = \bigcup_{n=1}^{N} V_n$ be a finite union of compact sets, and let $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ be an arbitrary open cover of $\mathcal{V}$. Consider an arbitrary compact set V_n in our union. Since V_n compact, then there exists a finite set $U_n = \{U_{n,1}, \ldots, U_{n,m(n)} : U_{n,i} \in \mathcal{U}, \forall i = 1, \ldots, m(n)\}$ such that $V_n \subset \bigcup_{i=1}^{m(n)} U_{n,i}$. Then $\{U_{n,i} : i = 1, \ldots, m(n)\}_{n=1}^{N}$ is a finite subcover of $\mathcal{V}$.

Problem 5 (14 points). Vector spaces:

True/False. Determine whether the following sets are vector spaces. If they are, prove it. If not, demonstrate why.

a) The set of all positive definite $n \times n$ matrices.

Ans: False. Consider the matrix $[1]$. Multiplying this matrix by the scalar -1 provides the matrix $[-1]$, which is not positive definite. Thus, the set is not closed under scalar multiplication.

b) For a fixed vector $v \in \mathbb{R}^n$, the set $W = \{A_{n \times n} : v \text{ is an eigenvector of } A\}$.

Ans: True. Consider $v \in \mathbb{R}^n$. Null vector: the $n \times n$ matrix that is populated with zeros has v as an eigenvector with an eigenvalue equal to 0.
Closed under vector addition: Let $A_1, A_2 \in W$. Then v is an eigenvector of A_1 and A_2. Let λ_1, λ_2 be the corresponding eigenvalues for each matrix, so $A_1 v = \lambda_1 v$ and $A_2 v = \lambda_2 v$. Note by

the linear properties of matrices, we have

$$A_1v + A_2v = \lambda_1 v + \lambda_2 v$$
$$(A_1 + A_2)v = (\lambda_1 + \lambda_2)v$$

Clearly $B = A_1 + A_2$ is an $n \times n$ matrix, and v is an eigenvector of B with the eigenvalue $\lambda_1 + \lambda_2$. Thus, the set is closed under vector addition. Closed under scalar multiplication: Let $A \in W$ with associated eigenvalue λ, and $r \in \mathbb{R}$. Then $Av = \lambda v \Rightarrow rAv = r\lambda v$ Then v is also an eigenvector of the matrix $B = rA$, with eigenvalue $r\lambda$. Thus, the set is closed under scalar multiplication.

Problem 6 (15 points). Vector subspaces:
Consider a fixed $(a, b, c) \in \mathbb{R}^3$ and let$S_{a,b,c} = \{(x, y, z) \in \mathbb{R}^3 : ax + by + cz = 0\}$

a) Show that $S_{a,b,c}$ is a vector subspace of $\mathbb{R}^3$

Ans: Null vector: $(0, 0, 0) \in S_{a,b,c}$ since $0a + 0b + 0c = 0$.
Closed under scalar multiplication: Consider any $(x, y, z) \in S_{a,b,c}$ and any $\lambda \in \mathbb{R}$. Note

$$ax + by + cz = 0$$
$$\Rightarrow \lambda(ax + by + cz) = \lambda 0$$
$$\Rightarrow a\lambda x + b\lambda y + c\lambda z = 0$$

Thus $\lambda * (x, y, z) \in S_{a,b,c}$.
Closed under vector addition: Consider two vectors (x_1, y_1, z_1), $(x_2, y_2, z_2) \in S_{a,b,c}$. Then

$$x_1a + y_1b + z_1c = 0,$$
$$x_2a + y_2b + z_2c = 0$$
$$\Rightarrow x_1a + x_2a + y_1b + y_2b + z_1c + z_2c = 0$$
$$\Rightarrow a(x_1 + x_2) + b(y_1 + y_2) + c(z_1 + z_2) = 0$$

Thus the set is closed under vector addition.

b) True or false: $S_{1,1,1} \cup S_{1,2,3}$ is a vector subspace of $\mathbb{R}^3$. If true, provide a proof. If false, show by counterexample that some property of a vector subspace is violated.

Ans: False. The set fails vector addition. Note that $(1, -1, 0) \in S_{1,1,1}$ and $(2, -1, 0) \in S_{1,2,3}$. But $(1, -1, 0) + (2, -1, 0) = (3, -2, 0)$. This vector is in neither $S_{1,1,1}$ nor in $S_{1,2,3}$ since

$$3 * 1 - 2 * 1 + 0 * 1 = 1 \neq 0 \;\Rightarrow (3, -2, 0)\text{not in } S_{1,1,1}$$
$$3 * 1 - 2 * 2 + 0 * 3 = -1 \neq 0 \;\Rightarrow (3, -2, 0)\text{not in } S_{1,2,3}$$

c) Consider the set $A_{1,2,1} = \{(x, y, z) \in \mathbb{R} : x + 2y + z = 1\}$. True or false: $A_{1,2,1} \cap S_{1,1,1}$ a vector subspace of $\mathbb{R}^3$. If true, provide a proof. If false, show by counterexample that some property of a vector subspace is violated.

Ans: False. Note that $(0, 0, 0) \notin A_{1,2,1}$ so $(0, 0, 0) \notin A_{1,1,1} \cap S_{1,1,1}$. In addition, the intersection is not closed under addition nor scalar multiplication.

Problem 7 (14 points). Bases:

For each of the following sets of sequences in $\mathbb{R}$, can you find a basis for it? If so, what is it? If not, why not?

a) $\{\{x_n\} : x_n \neq 0 \Leftrightarrow n \text{ is even}\}$

Ans: This set is not a vector space and therefore we cannot find a basis for it. Consider the sequences $\{0 - 1, 0, -1, 0, -1, .\}$ and $\{0, 1, 0, 1, 0, 1, \}$. Their sum is the null vector $\{0, 0, 0, 0, \}$, which is not in the set. Thus, the set is not closed under addition.

b) $\{\{x_n\} : x_n = 1 \Rightarrow n \text{ is odd}\}$

Ans: No. This set does is not a vector space. For example, it is not closed under scalar multiplication since the sequence $\{x_n\} = \{0, 1/2, 0, 0, 0, \dots\}$ is in the set, but $2 * \{x_n\} = \{0, 1, 0, 0, 0, \dots\}$ is not in the set. Thus, a basis cannot be defined on this set.