Solution Key

In-Class Midterm Exam: Tues Oct. 27th, 2009

There are a total of 100 points on the exam; 64 points for analysis and 36 points for linear algebra. There is also a bonus problem worth 10 additional points.

Analysis (64 pts)

1. (Total: 24 pts) Indicate whether each statement is true or false. If true, prove your claim. If false, give a counterexample.

   (a) (6 pts) Let $f : \mathbb{R} \to \mathbb{R}$. If $f(x_n) \geq a \in \mathbb{R}$ for all $n \in \mathbb{N}$, and $f(x_n) \to f(x)$ as $n \to \infty$, then $f(x) \geq a$.

   True. (Proof by contradiction) Suppose $a > f(x)$. Define $\delta = a - f(x) > 0$. We are given $f(x_n)$ converges to $f(x)$ so we know that $\forall \epsilon > 0$ then $\exists N \in \mathbb{N}$ such that $\forall n > N$ we have $|f(x_n) - f(x)| < \epsilon$. Now note $|f(x_n) - a + \delta| = |f(x_n) - f(x)|$. By definition $\delta > 0$ so $\forall n > N$ we know $|f(x_n) - a| < |f(x_n) - a + \delta| = |f(x_n) - f(x)| < \epsilon$. And $f(x_n)$ has two distinct limit points, a and $f(x)$. Contradiction, limit points are unique; therefore, our supposition is false, and $f(x) > a$.

   (b) (6 pts) If $x_n \in \mathbb{R}$ is a convergent sequence, then x_n is bounded below.

   True. (Direct proof) We are given x_n converges so that $\forall \epsilon > 0$ then $\exists N \in \mathbb{N}$ such that $\forall n > N$ $|x_n - x| < \epsilon$ for some x. Define $S_1 = \{x_n : n < N\}$ and let $M_1 = \min\{S_1\}$. Next define $S_2 = \{x_n : n > N\}$. Since we are given $|x_n - x| < \epsilon$ we know that $x_n > x - \epsilon$, $\forall n > N$. We have shown all elements of x_n are greater than $\min\{M_1, x - \epsilon\}$; therefore any convergent sequence x_n is bounded below.

   (c) (6 pts) If every convergent subsequence of x_n converges to x, then x_n converges to x.

   False. Counterexample: Define $x_n = n$ for n odd and $x_n = \frac{1}{n}$ for n even. Clearly every convergent subsequence goes to zero; however, x_n itself does not converge.

(d) (6 pts) Let $x_n \in X \subset \mathbb{R}$. If $x_n \to x$ as $n \to \infty$ then x is an accumulation point of X.

False. Counterexample: Define $x_n = 1$ and $X = \mathbb{N}$. Clearly, $x_n \to 1$. Also note that for $\epsilon < 1$ the ϵ-ball around 1 contains no other points in X, and so is not an accumulation point of X.

2. (Total: 16 pts) Let $x : \mathbb{N} \to \mathbb{R}$, and let $X = x(\mathbb{N})$ be the image of this mapping. Assume that $\{1/n : n \in \mathbb{N}\} \subset X$. Assume also that for infinitely many n in $\mathbb{N}$, $x(n) = 1$.

(a) (6 pts) Define $x_n = x(n)$. If the metric on $\mathbb{R}$ is the discrete metric, is it possible that x_n converges? If yes, then sketch how you would construct such a sequence. If no, prove that such a sequence does not exist.

Impossible. (Proof by contradiction) Suppose such a sequence does exist. Since we are in the discrete metric we know that in order for x_n to converge then for some $N \in \mathbb{N}$ we must have x_n be constant $\forall n > N$. This implies $\{1/n : n \in \mathbb{N}\} \subset \{n : n < N + 1, n \in \mathbb{N}\}$. Contradiction, and no such sequence exists.

(b) (10 pts) If the metric on $\mathbb{R}$ is the Euclidean metric, construct a subsequence of x_n that converges to zero. (Hint: For this part, your answer needs to meet Leo's highest standards of formality in order to get high marks. You may not assume *anything* more about the sequence other than exactly what's specified.)

Let $\tau : \mathbb{N} \to \mathbb{N}$, and define $\tau(m) = 1$ if m=1 and $\tau(m)$=min$\{A_m\}$ otherwise, where $A_m = \{n \in \mathbb{N} : n > \tau(m-1)$ s.t. $x_n < x_{\tau(m-1)}\}$. We want to show that $\forall m \in \mathbb{N}$ $\tau(m) < \tau(m+1)$. We begin with the base case of $m = 1$ so that $\tau(1) = 1$ and $\tau(2)$=min$\{A_2\}$. By the Well-Ordering Property (every set of natural numbers has a least element), as long as A_2 is non-empty then by construction we will have $\tau(1) < \tau(2)$. Since we know $\frac{1}{2} \in X$ then there must exist $x_n < x_{\tau(2-1)=1}$ and $1 < n$ implying A_2 is non-empty. Hence, for $m = 1$ we have $\tau(1) < \tau(2)$. Now assume that for $m \in \mathbb{N}$, $\tau(m) < \tau(m+1)$. We want to show that for $m \in \mathbb{N}$, $\tau(m+1) < \tau(m+2)$. By the Well-Ordering Property, as long as A_{m+2} is non-empty then by construction we will have $\tau(m+1) < \tau(m+2)$. By way of contradiction assume $A_{m+2} = \{\emptyset\}$. First note that $x_{\tau(m+1)}$ is of the form $\frac{1}{n}$ with $n \in \mathbb{N}$, without loss of generality let $x_{\tau(m+1)} = \frac{1}{n'}$. Also note that the set $B = \{\frac{1}{n} : n > n', n \in \mathbb{N}\}$ has infinitely many elements (by the Archimedian Property). Furthermore, $B \subset X$ where X is defined as above (the set of all elements in $\{x_n\}_{n=1}^{\infty}$). We assumed $A_{m+2} = \{\emptyset\}$ which implies no element of B lives in the tail of x_n so all the elements in B must live in the head of the sequence (where the head is the set of elements in x_n such that $n < \tau(m+1)$ and the tail is the set of elements in x_n such that $n \geq \tau(n+1)$). Contradiction, the head of x_n is a finite set and cannot contain the infinite set B. Hence, A_{m+2} is not empty and so there exists $n > \tau((m+1)-1)$ such that $x_n < x_{\tau((m+1)-1)}$. As $\tau(n)$ is a strictly increasing function mapping $\mathbb{N}$ to $\mathbb{N}$ we know $x_{\tau(n)}$ is a subsequence of x_n. Furthermore, when endowed with the Euclidean metric $x_{\tau(n)}$ converges to 0 as $n \to \infty$.

3. (Total: 24 pts) Let $\Psi(x) = \begin{cases} 1 & \text{if } x \leq 1 \\ [0,\,1] & \text{if } x = 1 \\ [0,\,x] & \text{if } x > 1 \text{ and } x \in \mathbb{Q} \\ (0,\,1] & \text{if } x > 1 \text{ and } x \in \mathbb{R} \sim \mathbb{Q} \end{cases}$

(a) (2 pts) Draw the graph of the correspondence.

(b) (4 pts) Let $O = (0, 2)$. What are the upper and lower inverse images of $\Psi(O)$?

$\overline{\Psi}^{-1}(O) = \{x : x \leq 1\} \bigcup \{x : x > 1, x \in \mathbb{R} \sim \mathbb{Q}\}$
$\underline{\Psi}^{-1}(O) = \mathbb{R}$

(c) (6 pts) Is Ψ u.h.c.? If so, briefly explain how you know. If not, establish this definitively.

No, pick any irrational x' in the domain greater than one. You can easily find a neighborhood of $\Psi(x')$, call it U, such that for any neighborhood V of x' in the domain then $\exists x'' \in V$ such that $\Psi(x'')$ is not a subset of U (note that this x'' is rational).

(d) (2 pts) If the lecture notes had included the notion of a bounded-valued correspondence analogous to the definitions of closed-valued and compact-valued correspondences then what would be the definition of a bounded-valued correspondence? Write down the definition.

Definition: Ψ is a *bounded-valued correspondence* if for each x in the domain, $\Psi(x)$ is a bounded set.

(e) (2 pts) Is Ψ a bounded-valued correspondence? If so, briefly explain how you know. If not, establish this definitively.

Yes, Ψ is bounded-valued. Why? Pick an x in the domain, and $\Psi(x)$ is bounded above by x and bounded below by zero.

(f) (2 pts) Does Ψ have a compact graph if we change the domain to $\mathbb{Q}$? If so, briefly explain how you know. If not, establish this definitively.

Ψ does not have a compact graph since it is unbounded (To prove this to yourself, suppose b is an upperbound of Graph($\Psi(\mathbb{R})$). Next find a rational number, say x''', in the domain greater than b and we have found $\Psi(x''') \subseteq$Graph($\Psi(\mathbb{R})$) where $\Psi(x''')$ contains elements greater than b; therefore, Graph($\Psi(\mathbb{R})$) is unbounded and so is not compact.

(g) (6 pts) Is Ψ l.h.c. if we restrict the domain to $\mathbb{R} \sim \{1\}$? If so, briefly explain how you know. If not, establish this definitively.

No, it is not l.h.c.. Applying the neighborhood definition of l.h.c. we pick any rational x greater than one from the domain, say $x = 2$. Consider the open set $G = (1.5, 2)$ in the range, and we observe $\Psi(2) \bigcap G \neq \emptyset$. Now in the domain pick any neighborhood V around 2; V contains an irrational number, say w. Note that $\Psi(w) = (0, 1]$; it is easy to see that $\Psi(w) \bigcap G = \emptyset$, and so Ψ does not satisfy the neighborhood formulation of l.h.c..

4. **Bonus** (Total: 10 pts) Let $\xi: \mathbb{R} \rightrightarrows \mathbb{R}$

$$\xi(x) = \begin{cases} [-\frac{1}{x}, \frac{1}{x}] \bigcap \{\mathbb{R} \sim \mathbb{Q}\} & \text{if } x > 0 \\ \mathbb{R} & \text{if } x \leq 0 \end{cases}$$

(a) (2 pts) Draw the graph of $\xi(x)$.

(b) (6 pts) Is ξ l.h.c.? If so, briefly explain how you know. If not, establish this definitively.

Yes, it is l.h.c.. Pick any x' in the domain. Pick any neighborhood G in the range such that $\xi(x') \bigcap G \neq \{\emptyset\}$. It is clear from the graph that there exists a neigborhood V of x' in the domain such that for any x in V we have $\xi(x) \bigcap G \neq \{\emptyset\}$. Since we picked arbitrary x' and arbitrary G, the result holds in general and ξ satisfies the neighborhood formulation of l.h.c.. In a similar fashion it is easy to verify that the sequential and inverse image formulations of l.h.c. hold.

(c) (2 pts) What are the boundary points of the graph of $\xi(\mathbb{R})$? That is, what is the set bd(Graph($\xi(\mathbb{R})$)) equal to?

bd(Graph($\xi(\mathbb{R})$))= $\{(x, y) \in \mathbb{R}^2 : x \geq 0, \& \, y \in [-\frac{1}{x}, \frac{1}{x}]\}$

Linear Algebra (36 points)

5. (Total: 12 pts) Indicate whether the following are “necessary” or “sufficient” or “necessary & sufficient” or “neither” for an $m \times n$ matrix A to be invertible.

   (a) (2 pts) n=m

   Necessary.

   (b) (2 pts) A has no column vectors that are the zero vector.

   Necessary.

   (c) (2 pts) det(A)$\neq 0$

   Necessary and sufficient.

   (d) (2 pts) row rank(A)$\geq$column rank(A)

   Necessary.

   (e) (2 pts) A is symmetric and has no zero diagonal elements.

   Neither.

   (f) (2 pts) The column vectors form a basis for $\mathbb{R}^n$.

   Necessary and sufficient.

6. (Total: 12 pts) Indicate whether each statement is true or false. If true, prove your claim. If false, give a counterexample.

(a) (4 pts) If u_1, u_2, and u_3 span V, then $\dim(V)$ =3.

False, let $u_1 = (1, 0)$, $u_2 = (0, 1)$, and $u_1 = (1, 1)$; these vectors span a vector space V with $\dim(V)$= 2.

(b) (4 pts) If A is a 4×6 matrix, and rank(A)=4, then any four column vectors of A are linearly independent.

$$\text{False, let } A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

(c) (4 pts) Suppose that u_1, u_2, u_3, and w all belong to some vector space V. If u_1, u_2, and u_3 are linearly independent, then u_1, u_2, u_3, and w are linearly dependent.

$$\text{False, let } [u_1,\, u_2,\, u_3,\, w] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

7. (Total: 6 pts) Let U and W be subspaces of a vector space V. Show that the intersection $U \bigcap W$ is also a subspace of V.

U and W are both subspaces of a vector space and so both must contain the zero vector; hence, $U \bigcap W$ is non-empty. It remains to show that $U \bigcap W$ is closed under addition and proportionality. Pick any two v_1, v_2 in $U \bigcap W$. For any α, $\beta \in \mathbb{R}$ then αv_1 and βv_2 belong to U which implies $\alpha v_1 + \beta v_2 \in U$. Likewise, for any α, $\beta \in \mathbb{R}$ then αv_1 and βv_2 belong to W which implies $\alpha v_1 + \beta v_2 \in W$. It follows that for any α, $\beta \in \mathbb{R}$ and for any v_1, $v_2 \in U \bigcap W$ then $\alpha v_1 + \beta v_2 \in U \bigcap W$. Thus, if U and W are subspaces of a vector space V, then the intersection $U \bigcap W$ is also a subspace of V.

8. (Total: 6 pts) Consider two 2×2 symmetric matrices M^a and M^b. Suppose that the set of eigenvectors for both matrices are identical. Suppose further that for $i = a, b$,

the eigenvalues of matrix M^i are $\lambda_1^i > 0 > \lambda_2^i$, where $\lambda_1^a = \lambda_1^b$ and $|\lambda_2^b| < |\lambda_2^a|$. For $i = a, b$, let $P^i = \{x \in \text{ unit circle: s.t. } x'M^i x > 0\}$. Which of the following statements is true? Justify your answer.

(a) $P^a \subsetneq P^b$

(b) $P^b \subsetneq P^a$

(c) $P^a = P^b$

(d) none of the above

(The notation $U \subsetneq V$ means that U is contained in V but is not equal to V.)

(a) $P^a \subsetneq P^b$ is true, the rest are false.