In class midterm Exam
Thu Oct 2

Please do SIX out of the first SEVEN questions. Question 8 on hemi-continuity is required. You can attempt the remaining question for extra credit. Thus, there are a total of 100 points for the exam $(6 \times 14 + 16)$ **and a possible 14 points of extra credit.**

Rules, regulations and advice

a) If we want you to use a specific metric for a question, we'll state this explicitly. Otherwise, your answer should hold for any metric.
b) You don't have to answer the questions in order, but please number each of your answers carefully.
c) Some questions are significantly easier, and quicker to complete, than others. Get the easy/quick ones out of the way first before trying the longer/harder ones.
d) Use good judgement when it comes to deciding what to prove and what not to. For example it would be a really bad judgement call to take precious time to *prove* that the set of integers $\mathbb{Z}$ contains no accumulation points.
e) It's very hard to get partial credit for a question unless you write at least something down. So unless you are very fast, *don't obsess*. If you are pressed for time, it's almost always better to give quick-and-dirty answer to more questions than beautiful, perfect answers to fewer. Once you finish the exam, you can always go back at the end and improve on the quick-and-dirty answers. In particular, if a question has a yes/no answer, write down the answer, then come back and justify it later.

Problem 1 (14 points). Accumulation points:

Let $\{a_n\}$ be a sequence in some set $A \subset \mathbb{R}$. Prove that if $lim\, a_n = x$, but $a_n \neq x$ for all n, then x is an accumulation point of the set A.

Problem 2 (14 points). Open and closed sets:

For this question, use a Euclidean metric. Let $B = \{\frac{(-1)^n n}{n+1} \ : \ n = 1, 2, 3, ...\}$.

a) Does the set B in $\mathbb{R}$ have any accumulation points? If so, what are they?
b) Is B a closed subset of $\mathbb{R}$?
c) Is B an open subset of $\mathbb{R}$?

Problem 3 (14 points). Limits:

For this problem, consider an arbitrary universe $\mathbf{X}$ and an arbitrary metric d defined on $\mathbf{X} \times \mathbf{X}$. Let $\{x_n\}$ be a sequence in some set X. Prove that if $\{x_n\}$ has a limit, then this limit is unique.

Problem 4 (14 points). Sequences:

Give an example of each of the following, or if no such example exists, explain why not:

a) Sequences $\{x_n\}$ and $\{y_n\}$ which do not converge, but whose sum $\{x_n + y_n\}$ converges.
b) An unbounded sequence $\{x_n\}$ and a convergent sequence $\{y_n\}$, where $\{x_n - y_n\}$ is bounded.
c) A convergent sequence $\{x_n\}$, where $\{1/x_n\}$ does not converge.
d) Sequences $\{x_n\}$ and$\{y_n\}$ where $\{x_n\}$ converges, $\{y_n\}$ does not converge, and the sum $\{x_n + y_n\}$ converges.
e) Two sequences $\{x_n\}$ and $\{y_n\}$, where $\{x_n y_n\}$ and $\{x_n\}$ converge, but $\{y_n\}$ does not.

Problem 5 (14 points). Boundedness:

Show that if f is continuous on $[a, b]$ in $\mathbb{R}$ with $f(x) > 0$ for all $x \in [a, b]$, then $\frac{1}{f}$ is bounded on $[a, b]$

Problem 6 (14 points). Continuity:

For this question, use an Euclidean metric. Show that any function f whose domain is $\mathbb{Z}$ (the integers) will be continuous at every point in its domain.

Problem 7 (14 points). More continuity:

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous on $\mathbb{R}$ and let $K = \{x \in \mathbb{R} : f(x) = 0\}$. Show that K is closed in $\mathbb{R}$.

Problem 8 (16 points). Hemi-continuity:

Consider the following two correspondences mapping $X = [-1, 1]$ to $Y = [0, 4\pi]$:
$\xi(x) = \{y \in Y : \sin(y) = x\}$ and $\phi(x) = \{y \in Y : \cos(y) = x\}$.

a) Sketch a graph of each correspondence.
b) Are either or both of these correspondences upper-hemi-continuous. If so, *briefly* explain how you know. If not, establish this definitively.
c) Are either or both of these correspondences lower-hemi-continuous. If so, *briefly* explain how you know. We don't expect a formal proof, just an indication that you know what's going on. If not, establish this definitively.