In class midterm Exam - Answer key

Problem 1 (14 points). Accumulation points:

Let $\{a_n\}$ be a sequence in some set $A \subset \mathbb{R}$. Prove that if $lim\, a_n = x$, but $a_n \neq x$ for all n, then x is an accumulation point of the set A.

Ans: x is an accumulation point of a set A if for every $\epsilon > 0$, the ball $B(x, \epsilon)$ contains a point $a \in A$ such that $a \neq x$, Fix $\epsilon > 0$. Since $(a_n) \to x$, there exists N such that for $n > N$, $a_n \in B(x, \epsilon)$. Moreover, by assumption $a_n \neq x$. Hence, x satisfies the definition of an accumulation point.

Problem 2 (14 points). Open and closed sets:

For this question, use a Euclidean metric. Let $B = \{\frac{(-1)^n n}{n+1} \ : \ n = 1, 2, 3, ...\}$.

a) Does the set B in $\mathbb{R}$ have any accumulation points? If so, what are they?

Ans: For any $\epsilon > 0$, there exists N such that for $n > N$, if n is *even* then $\frac{n}{n+1} \in (1-\epsilon, 1)$, while if n is odd, then while if n is odd, then $-\frac{n}{n+1} \in (-1, -(1-\epsilon))$. Thus, B has two accumulation points, 1 and -1.

b) Is B a closed subset of $\mathbb{R}$?

Ans: No, since 1 is an accumulation point of B but is not contained in B.

c) Is B an open subset of $\mathbb{R}$?

Ans: No, since $1/2$ is an element of B but there is no open set around $1/2$ which is contained in B.

Problem 3 (14 points). Limits:

For this problem, consider an arbitrary universe $\mathbf{X}$ and an arbitrary metric d defined on $\mathbf{X} \times \mathbf{X}$. Let $\{x_n\}$ be a sequence in some set X. Prove that if $\{x_n\}$ has a limit, then this limit is unique.

Ans: Let x be the limit of $\{x_n\}$ and let y be any other element of X. Since $y \neq x$, there exists $\epsilon > 0$ such that $d(y, x) > 2\epsilon$. Since $\{x_n\}$ converges to x, there exists N such that for all $n > N$, $d(x_n, x) < \epsilon$. Hence by the triangle inequality, $d(x_n, y) > \epsilon$. Hence $\{x_n\}$ does not converge to y

Problem 4 (14 points). Sequences:

Give an example of each of the following, or if no such example exists, explain why not:

a) Sequences $\{x_n\}$ and $\{y_n\}$ which do not converge, but whose sum $\{x_n + y_n\}$ converges.

Ans: Let $x_n = n$ and $y_n = -n$. Clearly, neither $\{x_n\}$ nor $\{y_n\}$ converge, but $x_n + y_n = 0$ is a constant sequence, which converges to 0

b) An unbounded sequence $\{x_n\}$ and a convergent sequence $\{y_n\}$, where $\{x_n - y_n\}$ is bounded.

Ans: Such a sequence cannot exist. To see this, let $\{y_n\}$ converge to y, so that for some N and all $n > N$, $y_n > y - 1$. Now assume w.l.o.g. that $\{x_n\}$ is unbounded above, i.e., for all $M > N$ there exists $n > M$, such that $x_n > M$. Hence $x_n + y_n > x_n + y - 1 > M + y - 1$, establishing that $\{x_n + y_n\}$ is unbounded.

c) A convergent sequence $\{x_n\}$, where $\{1/x_n\}$ does not converge.

Ans: Let $x_n = 1/n$. In this case, $1/x_n = n$, which does not converge.

d) Sequences $\{x_n\}$ and$\{y_n\}$ where $\{x_n\}$ converges, $\{y_n\}$ does not converge, and the sum $\{x_n + y_n\}$ converges.

Ans: Such a sequence cannot exist. To see this, let $\{x_n\}$ converge to x and choose $y \in \mathbb{R}$ arbitrarily. Since $\{y_n\}$ does not converge, there exists $\epsilon > 0$ such that for all n, there exists $k^n > n$ such that $d(y_{k^n}, y) > 2\epsilon$, and hence $d(x + y_{k^n}, x + y) > 2\epsilon$, while $d(x_{k^n}, x) < \epsilon$, and hence $d(x_{k^n} + y_{k^n}, x + y_{k^n}) < \epsilon$, The triangle inequality now implies

$$2\epsilon \quad < \quad d(x + y_{n^k}, x + y) \quad \leq \quad d(x + y_{n^k}, x_{n^k} + y_{n^k}) \quad + \quad d(x_{n^k} + y_{n^k}, x + y)$$

so that

$$\epsilon \quad < \quad 2\epsilon - d(x_{n^k} + y_{n^k}, x + y_{n^k}) \quad < \quad d(x_{n^k} + y_{n^k}, x + y)$$

Conclude that $\{x_n + y_n\}$ does not converge to $x + y$. Since y was chosen arbitrarily, we have established that the sequence has no limit point in $\mathbb{R}$.

e) Two sequences $\{x_n\}$ and $\{y_n\}$, where $\{x_n y_n\}$ and $\{x_n\}$ converge, but $\{y_n\}$ does not.

Ans: Let $x_n = 0$ and $y_n = n$. Obviously, $x_n = x_n y_n = 0$, so that $\{x_n\}$ and $\{x_n y_n\}$ both converge.

Problem 5 (14 points). Boundedness:

Show that if f is continuous on $[a, b]$ in $\mathbb{R}$ with $f(x) > 0$ for all $x \in [a, b]$, then $\frac{1}{f}$ is bounded on $[a, b]$

Ans: Obviously, f is bounded below by zero. Since f is continuous and $[a, b]$ is compact, it follows from Weierstrass's theorem that f attains a minimum on $[a, b]$, hence $\frac{1}{f}$ attains a maximum on $[a, b]$. This maximum is an upper bound on f on $[a, b]$.

Problem 6 (14 points). Continuity:

For this question, use an Euclidean metric. Show that any function f whose domain is $\mathbb{Z}$ (the integers) will be continuous at every point in its domain.

Ans: To establish that f is continuous, we need to show that if a sequence $\{x_n\}$ in $\mathbb{Z}$ converges to x, then $\{f(x_n)\}$ converges to $f(x)$. Since the set $\mathbb{Z}$ is a discrete set, i.e., it contains no accumulation points, it follows that there exists N such that if $\{x_n\}$ converge to x, then for $n > N$, $x_n = x$. Hence $f(x_n) = f(x)$, so that $\{f(x_n)\}$ converges to $f(x)$. Hence f is continuous.

Problem 7 (14 points). More continuity:
Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous on $\mathbb{R}$ and let $K = \{x \in \mathbb{R} : f(x) = 0\}$. Show that K is closed in $\mathbb{R}$.

Ans: Let x^n be a sequence in K that converges to $\bar{x}$. To show that K is closed, we need to show that $\bar{x} \in K$. Since $f(x^n) = 0$ for all n, we have $\lim_n f(x^n) = 0$. But since f is continuous, $\lim_n f(x^n) = f(\bar{x})$. Hence $f(\bar{x}) = 0$ verifying that $\bar{x} \in K$.

Problem 8 (16 points). Hemi-continuity:
Consider the following two correspondences mapping $X = [-1, 1]$ to $Y = [0, 4\pi]$:
$\xi(x) = \{y \in Y : \sin(y) = x\}$ and $\phi(x) = \{y \in Y : \cos(y) = x\}$.

a) Sketch a graph of each correspondence.

Ans: See Figure 1 below.

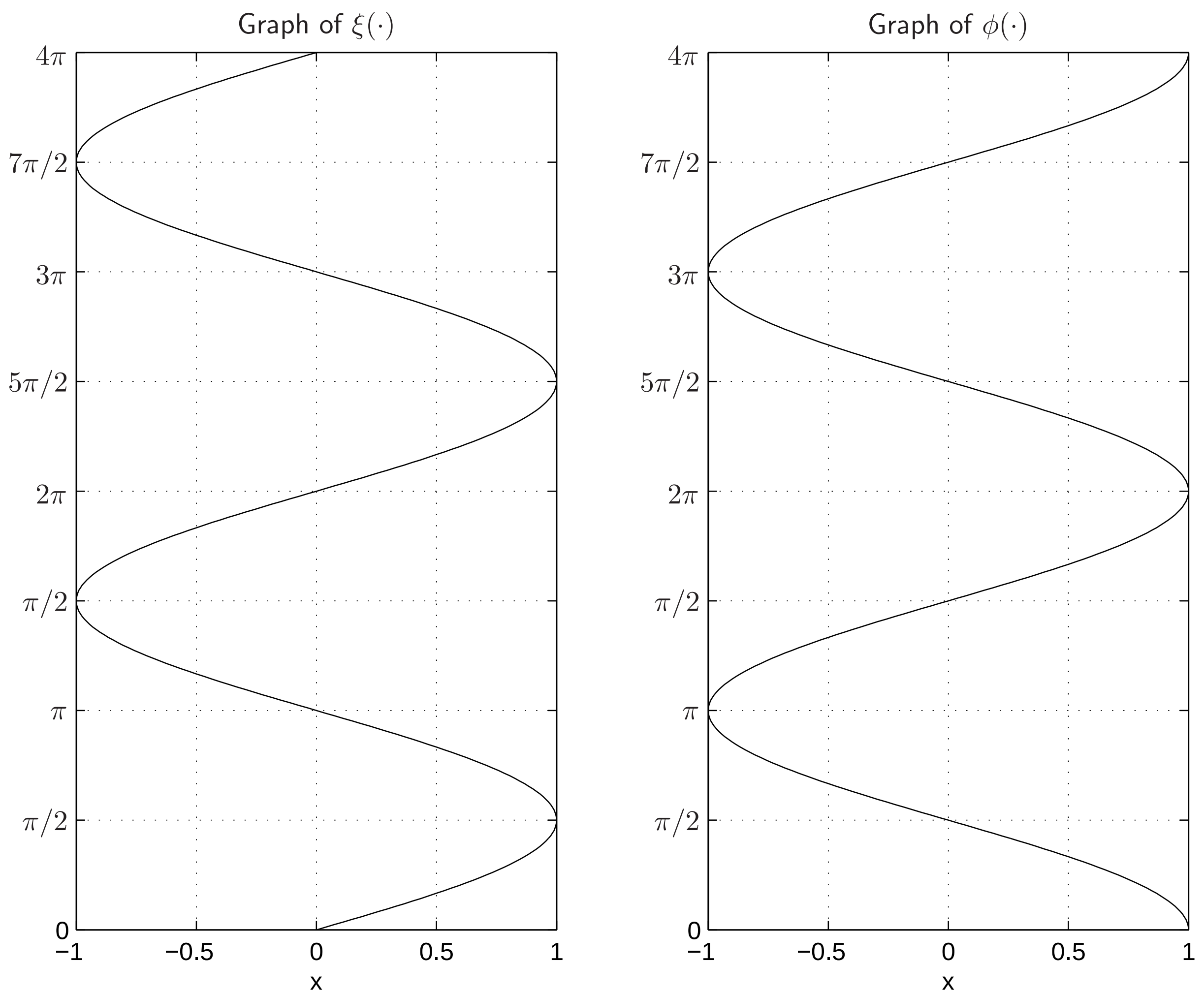

FIGURE 1. Graphs of ξ and ϕ.

b) Are either or both of these correspondences upper-hemi-continuous. If so, *briefly* explain how you know. If not, establish this definitively.

Ans: Both are u.h.c. To prove this, note that both graphs are closed valued. Also the range of both graphs is compact, i.e., $[0, 4\pi]$. Moreover both correspondences have closed graphs. Hence they are u.h.c.

c) Are either or both of these correspondences lower-hemi-continuous. If so, *briefly* explain how you know. We don't expect a formal proof, just an indication that you know what's going on. If not, establish this definitively.

Ans: ξ is not lower hemi-continuous. To show this, consider a small neighborhood O of 4π. The lower inverse image of O under ξ is $(-\epsilon, 0]$ for some $\epsilon > 0$. This set is not closed. On the other hand, ϕ *is* lower hemi-continuous. To see this, we merely note that, unlike the example in the 6th Analysis lecture, the 'vertical points' of the graph in this case are all at the boundary of the domain. So, for example consider a small neighborhood O of 2π. In this case, the lower inverse image of O under ϕ is the *open* set $(1 - \epsilon, 1]$ for some ϵ.