Midterm Exam

Due date: Mon Oct 22, noon. (Leo's mailbox in 207 Giannini)

There are 7 questions.
Each questionquestion is worth 14 points, except for #6, which is worth 16, for a total of 100 points.

Rules, regulations and advice

a) You must work on this exam by yourself. You are not allowed to talk to anybody about it or exchange notes (not even old class notes) with anybody during the exam period. You can use *your own* notes, the lecture notes, the answer keys to the problem sets, as well as books such as Simon & Blume.

b) Please write up the answers neatly; otherwise points may be subtracted because your answer is not legible.

c) Conciseness is a virtue of proofs. We will **definitely** take points off for excessively long proofs. It is just not fair to give somebody the same points for writing a proof covering three pages that other people did perfectly in four lines. (**None of the questions requires a proof that is long, so if your answer is getting really long and messy, you must be missing something**)

d) Some questions, or parts of questions might require a little trick which might be difficult to see. So, please work on the easy questions, and the easy parts of questions that have hard parts, first and save the difficult questions for the end. Each question has a "theme," so if you are having trouble with figuring out what the question is getting at, try a different part to see if you can pick up the theme.

e) If you believe a question needs clarification, or even worse, is incorrect (I hope this doesn't happen) send me an email, and copy Rosangela. Remember to put ARE211 in the subject line, to defeat the spam police. If I consider it appropriate, I will reply to the whole class so everybody gets the same response. You might want to check your email periodically, just to see if there's any correspondence. **Please note well:**
   i) in past years, more than half the email questions I received were undecipherable, and had to be returned for further clarification. This was a *major* pain. Hence please follow the following simple instructions for submitting questions.
   ii) **always** specify the **precise** part of the question to which you are referring.
   iii) your questions will (should) always relate to very fine details of the question. So if you think I've mis-worded a question (which, by the way, is unlikely, though possible) specify in a full sentence the re-wording you think is correct. Write mathematical symbols in words, i.e., if you want to write $x \in B(x, \delta) \subset X$, write "x in B(x,delta) subset X" and I'll know what you mean.

Problem 1: Continuity. (14 points)

In the first two parts of this problem, determine whether or not the specified functions are continuous. If they are, prove it. If they are not, identify the subset X in the domain of the function such that for every $x \in X$, f is continuous at x.

a)

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

b)

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1/q & \text{if } x = p/q \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

c) To do this part, you need the formal definition of the derivative of a function. Let $g : \mathbb{R} \to \mathbb{R}$ and pick $x \in \mathbb{R}$. The derivative of g at.x is defined as:
$g'(x) = \lim_{h \to 0} \frac{g(x+h)-g(x)}{h}, (h \in \mathbb{R})$. Now define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} x^n \sin(x^{-m}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

where n and m are natural numbers. Identify conditions on n and m that are necessary and sufficient for the following properties:

i) g is differentiable at 0.

ii) $g'(\cdot)$ is continuous at 0.

Problem 2: Metrics. (14 points)

a) Let x and y be two differentiable real valued functions. Is $d(x, y) = \int_{-\infty}^{\infty} |x(t) - y(t)|\, dt$ a metric? if it is, prove it. If not, explain why and provide a counter-example.

b) Let $f : \mathbb{R} \to \mathbb{R}$ and let $f^n(x)$ denote the n-th derivative of the function f with respect to x. For any point $x \in \mathbb{R}$, let $y_x^n = f^n(x)$. Find conditions on f so that $\{y_x^n\}$ converges in the Pythagorean metric for every $x \in \mathbb{R}$. Note that there are lots of sufficient conditions, many of them silly (e.g., a sufficient condition is that f is the zero function). To discourage silly answers like this, we're going to give you more marks, the larger is the set of functions that satisfies your sufficient condition.

Problem 3: More on metrics. (14 points)

a) Let for $i = 1, ...n$, let d^i be a metric on $\mathbb{R}$. and define $\mathbf{d} = (d^1, ..., d^i ..., d^n)$. Now for some function f, define $d^f(x, y) = f(\mathbf{d}(x, y))$. Is $d^f(x, y)$ a metric? If so, prove it. If not, provide sufficient conditions on f for it to be a metric. Once again, the weaker the set of conditions you provide (i.e., the more f's that satisfy the conditions), the more points you get.

b) Determine if either, neither or both $min(d_1, d_2)$ or $max(d_1, d_2)$ are metrics. If they are prove it. If not, provide a counter example.

c) Let

$$d(x,y) = \begin{cases} d^1(x,y) & \text{if } x > y \\ d^2(x,y) & \text{if } x \leq y \end{cases}$$

where $d^1(x,y)$ and $d^2(x,y)$ are metrics. Demonstrate that $d(x,y)$ is not a metric. A function very like d is, however, a metric. This function, call it ρ is defined almost identically to d, except that a small number of the symbols that define d (somewhere between 1 and 4 of them) have to be changed. Identify which symbols need to be changed, and verify that the newly defined function is indeed a metric.

Problem 4: Basic Analysis. (14 points)

Let $S = \{(x,y,z)\,|x = sin\phi\cos\theta, y = sin\phi\sin\theta, z = cos\phi, \phi > 0, \theta > 0\}$
Hint: this problem is much easier if you use "polar coordinates." Check wikipedia or any other source for a definition of polar coordinates.

a) Let $f(\mathbf{x},\mathbf{y}) = \varphi(\mathbf{x},\mathbf{y})$ where $\mathbf{x}$ and $\mathbf{y}$ belong to S and $\varphi(\mathbf{x},\mathbf{y})$ is the minimum non negative angle between the two vectors measured in radians. 2Π radians equal 360 degrees. Define an ϵ ball on S in this metric.

b) Let $\{x_n\} = \{\sin(\pi/2^n + \pi n/2)\cos(\pi n), \sin(\pi/2^n + \pi n/2)\sin(\pi n), \cos(\pi/2^n + \pi n/2)\}$
Use the pythagorean metric. Show that this sequence is not a Cauchy sequence.

c) Construct a subsequence of the sequence $\{\mathbf{x}_n\}$, defined in the previous part, that is Cauchy.

d) Determine if S is an open or a closed set in $\mathbb{R}^3$

e) Construct a new set S' by adding a new parameter ρ to the definition of S. Give conditions on this new parameter so that S' is open but for which all points in S are boundary points.

f) Is S a bounded set? Is S^c a bounded set?

Problem 5: Hemi-continuity. (14 points)

a) Let $\psi : \mathbb{R}^2 \twoheadrightarrow \mathbb{R}^2$ be defined as: $\psi(x,y) = (z,w)$, where
$$z = \begin{cases} [-1,1] & \text{if } x+y < 1 \\ 3/2 & \text{elsewhere} \end{cases} \quad and \quad w = \begin{cases} [-1,1] & \text{if } x+y < 1 \\ 2/3 & \text{elsewhere} \end{cases}$$
Is this correspondence upper hemi continuous? Lower hemi continuous? Continuous?

b) Let $\phi : \mathbb{R}^2 \twoheadrightarrow \mathbb{R}^2$ be defined as: $\phi(x,y) = (z,w)$, where
$$z = \begin{cases} [-1,1] & \text{if } x+y \geq 1 \\ 3/2 & \text{elsewhere} \end{cases} \quad and \quad w = \begin{cases} [-1,1] & \text{if } x+y > 1 \\ 2/3 & \text{elsewhere} \end{cases}$$

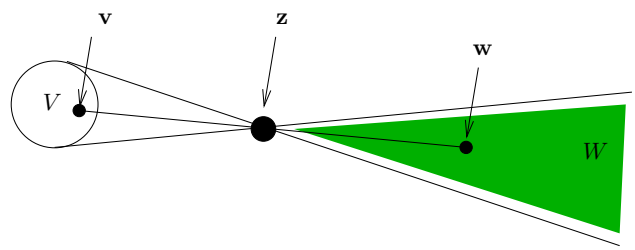

FIGURE 1. The set W is the shaded area on the right

Is this correspondence upper hemi continuous? Lower hemi continuous? Continuous?

c) Let $\chi : \mathbb{R}^2 \twoheadrightarrow \mathbb{R}^2$ be defined as: $\chi(x, y) = (z, w)$, where
$z = \begin{cases} 0 & \text{if } (x,y)=(0,0) \\ 1 & \text{elsewhere} \end{cases}$ *and* $w = \begin{cases} 0 & \text{if } (x,y)=(0,0) \\ -1 & \text{elsewhere} \end{cases}$
Is this correspondence upper hemi continuous? Lower hemi continuous? Continuous?

Problem 6: <u>Strict Quasi convexity</u>. (16 points)

a) (This is a little lemma that will help you prove the main result, below.)
Let V be an open set and fix $\mathbf{z} \in \mathbb{R}^n$. Let

$$W = \{\mathbf{w} \in \mathbb{R}^n : \mathbf{w} = \alpha\mathbf{z} + (1-\alpha)\mathbf{v}, \text{ for some } \mathbf{v} \in V \text{ and } \alpha > 1\}.$$

(See Fig. 1). Prove that W is an open set.

b) A continuous function $f : X \to \mathbb{R}$ is defined to be *strictly quasi-convex* if it satisfies the property (A), where

$$\forall \mathbf{x}, \mathbf{y} \in X \text{ s.t.} f(\mathbf{y}) \leq f(\mathbf{x}), \forall \lambda \in (0,1), f(\lambda\mathbf{y} + (1-\lambda)\mathbf{x}) < f(\mathbf{x}). \tag{A}$$

Show that condition (A) defining strict quasi-convexity is satisfied if and only if conditions (B) and (C) are satisfied, where

$$\text{every lower contour set of } f \text{ is a strictly convex set} \tag{B}$$

$$\text{every level set of } f \text{ has an empty interior} \tag{C}$$

Proving that (B) and (C) imply (A) is a bit tricky. So I suggest that you might consider trying to prove that $C \cap \neg A \Longrightarrow \neg B$, by proceeding along the following lines.

i) Find a point $\mathbf{z}$ on the line segment joining $\mathbf{x}$ and $\mathbf{y}$, where $f(\mathbf{y}) \leq f(\mathbf{x})$, that violates condition (A).

ii) There are now two mutually exclusive possibilities
   (i) x belongs to the interior of the lower contour set corresponding to $f(\mathbf{x})$.
   (ii) x doesn't belong to the interior of the lower contour set corresponding to $f(\mathbf{x})$, violating condition B.

   If the latter case holds, then you are done. Assume therefore that the former case applies.

iii) Now apply property (C) to the point $\mathbf{z}$, use your answer to the first part of this question, plus the continuity of f, use property (C) a second time, and conclude that for some $\alpha < f(\mathbf{x})$, the lower contour set of f corresponding to α is not a convex set.

Problem 7: <u>Kuhn Tucker</u>. (14 points)

For $c \in \mathbb{R}_{++}$, consider the problem

$$\max x^{1/2}y^{1/2}$$

$$x^2 + y^2 = 1 \tag{1}$$

$$cx + \sqrt{2}y \leq 2 \tag{2}$$

$$x \geq 0 \tag{3}$$

$$y \geq 0 \tag{4}$$

a) Write this problem in the standard format, i.e., $\max f(\mathbf{x})$ s.t. $\mathbf{g}(\mathbf{x}) \leq \mathbf{b}$.

b) Identify conditions on c, if such conditions exist, such that
   i) the constraint (2) is neither binding nor satisfied with equality at the solution to the problem.
   ii) the constraint (2) is satisfied with equality at the solution to the problem but is not binding
   iii) the constraint (2) is binding.

   In each of these cases graph the problem, *clearly* indicating the both the constraint set and the solution to the problem. Then provide the analytic solution. In each case, specify which of the other constraints are binding.

c) For the case in which constraint (2) is binding, indicate how the degree of bindingness (i.e., the magnitude of the Lagrangian multiplier) changes as c changes. Ideally, support your answer graphically.