Midterm Exam

Due date: Mon Oct 23, '06, noon. (Howard's mailbox in Giannini)

There are 6 questions

Rules, regulations and advice

a) You must work on this exam by yourself. You are not allowed to talk to anybody about it or exchange notes (not even old class notes) with anybody during the exam period. You can use *your own* notes, the lecture notes, the answer keys to the problem sets, as well as books such as Simon & Blume.

b) Please write up the answers neatly; otherwise points may be subtracted because your answer is not legible.

c) Conciseness is a virtue of proofs. We will **definitely** take points off for excessively long proofs. It is just not fair to give somebody the same points for writing a proof covering three pages that other people did perfectly in four lines. (**None of the questions requires a proof that is long, so if your answer is getting really long and messy, you must be missing something**)

d) Some questions, or parts of questions might require a little trick which might be difficult to see. So, please work on the easy questions, and the easy parts of questions that have hard parts, first and save the difficult questions for the end. Each question has a "theme," so if you are having trouble with figuring out what the question is getting at, try a different part to see if you can pick up the theme.

e) If you believe a question needs clarification, or even worse, is incorrect (I hope this doesn't happen) send me an email. If I consider it appropriate to reply, I will reply to the whole class so everybody gets the same response. You might want to check your email periodically, just to see if there's any correspondence. **Please note well:**
   i) last year, more than half the email questions I received were undecipherable, and had to be returned for further clarification. This was a *major* pain. Hence please follow the following simple instructions for submitting questions.
   ii) **always** specify the **precise** part of the question to which you are referring.
   iii) your questions will (should) always relate to very fine details of the question. So if you think I've mis-worded a question (which, by the way, is unlikely, though possible) specify in a full sentence the re-wording you think is correct. Write mathematical symbols in words, i.e., if you want to write $x \in B(x, \delta) \subset X$, write "x in B(x,delta) subset X" and I'll know what you mean.

Problem 1: Metrics and continuity. (10 points)

Let X be an arbitrary universe, let d be an arbitrary metric. For arbitrary $y \in X$, consider the function $f^y : X \Rightarrow \mathbb{R}^n$ defined by $f^y(\cdot) = d(\cdot, y)$.
Hint: To provide a counter-example to either of the conjectures below, you will need to specify an X, a metric d, an element $y \in X$, a sequence $\{x^n\}$ and a $\bar{x}$, and show that for these variables, the selected conjecture is false.

a) Prove, or provide a counter-example, to the following conjecture.

   Conjecture: The function f^y is continuous when d is the metric on the domain and the metric on $\mathbb{R}^n$ is the Pythagorian metric.

b) Prove, or provide a counter-example, to the following conjecture:

   Conjecture: ($\{ f^y(x^n)\}$ converges to $f^y(\bar{x})$ in the Pythagorian metric) implies ($\{x^n\}$ converges to $\bar{x}$ in the metric d).

Problem 2: Extrema. (20 points)

Let F denote the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$ and consider the following five properties:

P : f is strictly quasiconcave
Q : f is concave
R : f attains a global maximum
S : f attains a global minimum
T : f is bounded (that is, the image $f(\mathbb{R})$ is bounded above and bounded below.)

There are $2^5 = 32$ ways of combining these five properties (i.e., each can be satisfied or not). We have selected ten of these thirty-two. For each one of these ten, say whether or not there is an element of F that satisfies the combination. If your answer is "yes," provide an example; if it is "no," prove that the combination is internally inconsistent. (Note that you may not need to use *all five* properties in order to show internal inconsistency.) For example, for the combination (P Q ¬R ¬S ¬T) there does exist $f \in F$ satisfying these conditions: $f(x) = x$. The ten combinations are

a) (¬P ¬Q ¬R ¬S ¬T)
b) (¬P ¬Q R S ¬T)
c) (¬P ¬Q R S T)
d) (¬P Q ¬R ¬S T)
e) (¬P Q ¬R S ¬T)
f) (¬P Q R ¬S ¬T)
g) (¬P Q R S T)
h) (P ¬Q ¬R S ¬T)
i) (P ¬Q R ¬S T)
j) (P Q R ¬S T)

Problem 3: Metrics and Topology. (20 points)

The symbol $\mathbb{R}^n_{++}$ denotes the set of strictly positive vectors in $\mathbb{R}^n$.

a) The *max metric* on $\mathbb{R}^n$, written d^∞ is defined by, for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $d^\infty(\mathbf{x}, \mathbf{y}) = \max\{|x_i - y_i| : i = 1, ..., n\}$. Prove that d^∞ satisfies the triangle inequality. (**Help:** You may use the fact that the function $d^1 : \mathbb{R} \times \mathbb{R} \Rightarrow \mathbb{R}$, defined by $d^1(x, y) = |y - x|$ is a metric.)

b) Let F denote the set of all *bounded* functions mapping $\mathbb{R}$ to $\mathbb{R}$ and define the function d^{sup} on $F \times F$ as follows: for $f, g \in F$,

$$d^{\mathsf{sup}}(f, g) = \sup\{|f(z) - g(z)| : z \in \mathbb{R}\}$$

Prove that d^{sup} satisfies the triangle inequality on $F \times F$.

The remainder of this question relates to the function $\rho : (\mathbb{R}_+ \times \mathbb{R}_{++}) \times (\mathbb{R}_+ \times \mathbb{R}_{++}) \Rightarrow \mathbb{R}$, defined as follows: for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2_+$, with $x_2, y_2 > 0$:

$$\rho(\mathbf{x}, \mathbf{y}) = \sup\{|x_1 \cos(z/x_2) - y_1 \cos(z/y_2)| : z \in \mathbb{R}\}$$

c) Prove that ρ is a metric on $\mathbb{R}^2_{++}$.

d) Prove that ρ is not a metric on $(\mathbb{R}_+ \times \mathbb{R}_{++})$.

For the remainder of this question, the universe is $\mathbb{R}^2_{++}$.

e) Fix $\mathbf{x} = (1, 1)$ and $\mathbf{y} = (1, 2)$.
   i) On the same axes, plot the graphs of $x_1 \cos(\cdot/x_2)$ and $y_1 \cos(\cdot/y_2)$ on the interval $[0, 4\pi]$.
   ii) From your graphs, what is $\rho(\mathbf{x}, \mathbf{y})$?

A useful fact that is a little tricky to prove is:

$$\text{for all } \alpha, \gamma > 0, \text{ for all } \beta \neq \alpha, \rho((\gamma, \alpha), (\gamma, \beta)) \geq \gamma.$$

You can use this fact in the remaining parts of this question, without proving it.

f) Let $\mathbf{x} = (1, 1)$ and fix $\epsilon < 1$ and characterize (analytically, pictures optional) the set $B_\rho\left(\mathbf{x}, \epsilon | \mathbb{R}^2_{++}\right)$. .

g) For each of the following subsets of $\mathbb{R}^2_{++}$, specify whether the set is either
   - open
   - closed
   - both
   - neither

   If your answer implies that a set is *not* open, then write down a boundary point of the set which is contained in the set; if your answer implies that a set is *not* closed, then write down an element of the closure of the set which does not belong to the set itself.
   i) $\{1\} \times \mathbb{R}_{++}$ (i.e., $x_1 = 1$, $x_2 > 0$).
   ii) $\mathbb{R}_{++} \times \{1\}$ (i.e., $x_1 > 0$, $x_2 = 1$).
   iii) $[1, 2] \times [1, 2]$.
   iv) $(1, 2) \times (1, 2)$.
   v) $[1, 2] \times (1, 2)$.
   vi) $(1, 2) \times [1, 2]$.

h) One of the following statements is true, the other is false. Prove the one that is true, and provide a counter-example (plus brief explanation) of the one that is false.
   i) (a sequence $(\mathbf{x}^n) = (x_1^n, x_2^n)$ converges to $(\alpha, \beta) \in \mathbb{R}^2_{++}$ with respect to the metric ρ) implies ($\exists N \in \mathbb{N}$ such that $\forall n > N$, $x_1^n = \alpha$).
   ii) (a sequence $(\mathbf{x}^n) = (x_1^n, x_2^n)$ converges to $\bar{x} = (\alpha, \beta) \in \mathbb{R}^2_{++}$ with respect to the metric ρ) implies ($\exists N \in \mathbb{N}$ such that $\forall n > N$, $x_2^n = \beta$).

Problem 4: Concavity, convexity, quasi-concavity, quasi-convexity. (20 points)

a) Consider the function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = \begin{cases} 0 \text{ if } x \leq 0 \\ 1 \text{ if } x > 0 \end{cases}$. Is this function concave? If so, prove it. If not, prove that this function is not concave.

b) Can you construct a convex set $A \subset \mathbb{R}$ such that when the function f defined in the previous part is restricted to this domain, it is both discontinuous and concave? If so, provide an example of such an A and prove that f restricted to this domain is concave. (You don't need to actually *prove* that the function is discontinuous on A.) If not, prove that for any convex set A, if f restricted to A is discontinuous, then it cannot be concave.

Now let F denote the set of functions from $\mathbb{R}^n$ to $\mathbb{R}$.

c) Can a function in F be strictly-quasi-concave but not continuous? If so provide an example and *prove* that your example is strictly quasi-concave. If not, prove that strict quasi-concavity implies continuity; also provide a graphical example that illustrates your proof.

d) Can a function in F be quasi-concave but not continuous? If so provide an example. If so, provide an example and *prove* that your example is quasi-concave. If not, prove that quasi-concavity implies continuity; also provide a graphical example that illustrates your proof.

e) Give an example of a function in F that is strictly quasi-concave and strictly quasi-convex, but neither concave nor convex. You should give the formula for your function and then sketch it. Argue graphically that all the required properties are satisfied. (You don't need a formal proof, but you do need a convincing argument.)

f) For this part, let G denote the set of twice differentiable functions from $\mathbb{R}^1$ to $\mathbb{R}^1$ that are both strictly quasi-concave and strictly quasi-convex, but neither concave nor convex (i.e., not concave *and* not convex). Now consider the following conditions. (Reminder: the notation $g(\cdot) \geq 0$ means $\forall x, g(x) \geq 0$.)
   A1: *either* $f'(\cdot) \geq 0$ *or* $f'(\cdot) \leq 0$.
   A2: *either* $f'(\cdot) > 0$ *or* $f'(\cdot) < 0$.
   A3: f is strictly monotonic.[1]
   B1: $\exists x$ s.t. $f''(x) \geq 0$ *and* $\exists y$ s.t. $f''(y) \leq 0$
   B2: $\exists x$ s.t. $f''(x) > 0$ *and* $\exists y$ s.t. $f''(y) < 0$
   For each of the six combinations of "A" conditions and "B" conditions, (i.e., A1 & B1, A1 & B2, etc), state whether the pair of conditions is
   i) *necessary but not sufficient* for $f \in G$ (prove necessary, provide counterexample to sufficient).
   ii) *sufficient but not necessary* for $f \in G$ (prove sufficient, provide counterexample to necessary).
   iii) *necessary and sufficient* for $f \in G$ (prove necessary and sufficient).
   iv) *neither necessary nor sufficient* for $f \in G$ (provide counter examples to necessary and to sufficient).
   To answer this question, you may use the following results.
   T1: $f \in G$ is concave iff $\forall x\ f''(x) \leq 0$.
   T2: for $f \in G$ if $\forall x\ f''(x) < 0$, then f is strictly concave.
   T3: $f \in G$ is convex iff $-f$ is concave
   T4: $f \in G$ is strictly convex iff $-f$ is strictly concave

[1] strictly monotonic means *either* strictly increasing *or* strictly decreasing.

Problem 5: Necessary and sufficient conditions. (10 points)

Consider the function $f(x) = ax^3 + bx^2 + cx + d$, where $a, b, c, d \in \mathbb{R}$.

a) Give conditions on a, b, c, d that are necessary and sufficient for $f(\cdot)$ to obtain a global maximum on $\mathbb{R}$.
b) Give conditions on a, b, c, d that are necessary and sufficient for $f(\cdot)$ to obtain a unique global maximum on $\mathbb{R}$.
c) Give conditions on a, b, c, d that are necessary and sufficient for $f(\cdot)$ to obtain a global maximum on $\mathbb{R}_+$.

Problem 6: The Mantra. (20 points)

Consider the following nonlinear programming problem.

$$\begin{aligned} \text{maximize } f(\mathbf{x}) &= \cos(x_1 + 2x_2) s.t. \\ x_1 &\geq 0 \\ x_2 &\geq 0 \\ x_1 + x_2 &\geq \pi/4 \\ x_1 + x_2 &\leq 3\pi/4 \end{aligned}$$

The mantra (part 1) states that a necessary condition for $\mathbf{x}^*$ to solve this problem is that the gradient of f at $\mathbf{x}^*$ belongs to the nonnegative cone defined by the gradient vector(s) of the constraint(s) that are satisfied with equality at $\mathbf{x}^*$. (If there are no constraints satisfied with equality at $\mathbf{x}^*$, then the gradient of f at $\mathbf{x}^*$ must be zero.)

Hint: The gradient of f at $\mathbf{x} = (x_1, x_2)$ is $\begin{bmatrix} -1 \\ -2 \end{bmatrix} \sin(x_1 + 2x_2)$

a) Find all points in the constraint set at which the mantra (part 1) is satisfied.
   Hint: We suggest you check the three distinct cases in the following order.
   i) the interior of the constraint set
   ii) the four line segments (excluding the corners)
   iii) four corners
b) Find the solution to this problem and the $\mathbf{x}$-value(s) in the constraint set at which the maximum is attained.
c) Which of the constraints are binding?
d) At each of the four corners of the constraint set, and at the solution(s), indicate on your graph the gradients of the objective and the constraints that are satisfied with equality.