MIDTERM EXAM - ANSWER KEY

Problem 1: Metrics and continuity. (10 points)

Let X be an arbitrary universe, let d be an arbitrary metric. For arbitrary $y \in X$, consider the function $f^y : X \Rightarrow \mathbb{R}^n$ defined by $f^y(\cdot) = d(\cdot, y)$.
Hint: To provide a counter-example to either of the conjectures below, you will need to specify an X, a metric d, an element $y \in X$, a sequence $\{x^n\}$ and a $\bar{x}$, and show that for these variables, the selected conjecture is false.

a) Prove, or provide a counter-example, to the following conjecture.

Conjecture: The function f^y is continuous when d is the metric on the domain and the metric on $\mathbb{R}^n$ is the Pythagorian metric.

Ans: Fix a sequence $\{x^n\}$ that converges to $\bar{x} \in X$, w.r.t. the metric d. We need to show that $\{f^y(x^n)\}$ converges to $f^y(\bar{x})$. Fix $\epsilon > 0$ arbitrarily and pick $N \in \mathbb{N}$ such that for all $n > N$ $d(x^n, \bar{x}) < \epsilon$. From the triangle inequality, we have $d(x^n, y) \leq d(x^n, \bar{x}) + d(\bar{x}, y)$, so that $d(x^n, y) - d(\bar{x}, y) \leq d(x^n, \bar{x}) < \epsilon$. Hence $|f^y(x^n) - f^y(\bar{x})| = d(x^n, y) - d(\bar{x}, y) < \epsilon$.

b) Prove, or provide a counter-example, to the following conjecture:

Conjecture: ($\{ f^y(x^n)\}$ converges to $f^y(\bar{x})$ in the Pythagorian metric) implies ($\{x^n\}$ converges to $\bar{x}$ in the metric d).

Ans: Let $X = \mathbb{R}$, d be the Pythagorian metric and $y = 0$. Let $x^n = (-1)^n$ and let $\bar{x} = 1$. For each n, $f^y(x^n) = d(x^n, 0) = 1$. Also $f^y(\bar{x}) = d(\bar{x}, 0) = 1$. Therefore, the sequence $\{f^y(x^n)\}$ trivially converges to $f^y(\bar{x})$. However, the sequence $\{x^n\}$ does not converge to 1.

Problem 2: Extrema. (20 points)

Let F denote the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$ and consider the following five properties:

P : f is strictly quasiconcave
Q : f is concave
R : f attains a global maximum
S : f attains a global minimum
T : f is bounded (that is, the image $f(\mathbb{R})$ is bounded above and bounded below.)

There are $2^5 = 32$ ways of combining these five properties (i.e., each can be satisfied or not). We have selected ten of these thirty-two. For each one of these ten, say whether or not there is an element of F that satisfies the combination. If your answer is "yes," provide an example; if it is "no," prove that the combination is internally inconsistent. (Note that you may not need to use *all five* properties in order to show internal inconsistency.) For example, for the combination (P Q ¬R ¬S ¬T) there does exist $f \in F$ satisfying these conditions: $f(x) = x$. The ten combinations are

a) (¬P ¬Q ¬R ¬S ¬T)

Ans: Yes. $f(x) = x(x + 1)(x - 1)$.

b) (¬P ¬Q R S ¬T)

Ans: No. R and S implies T. To see this, let f be maximized at $\bar{x}$ and minimized at $\underline{x}$ and pick $N \in \mathbb{N}$ such that $N > \max(|f(\bar{x})|, |f(\underline{x})|)$. But this means that f is bounded above by N and below by $-N$.

c) (¬P ¬Q R S T)

Ans: Yes. $f(x) = \begin{cases} 0 \text{ if } |x| > 1 \\ x(x+1)(x-1) \text{ if } |x| \leq 1 \end{cases}$.

d) (¬P Q ¬R ¬S T)

Ans: No. ¬P and Q implies R. Specifically, the condition ¬P implies that there exists x^*, y^*, z^* and t^* such that $z^* = t^*x^* + (1-t^*)y^*$ and $f(z^*) \leq \min\{f(x^*), f(y^*)\}$. The condition Q implies that for all x, y s.t. $x \neq y$ and all $t \in (0,1)$, $f(tx + (1-t)y) \geq tf(x) + (1-t)f(y)$. First suppose $f(x^*) \neq f(y^*)$ and, without loss of generality, let $f(x^*) < f(y^*)$, so that $\min\{f(x), f(y)\} = f(x^*)$. If this were case, then $f(z^*) \leq f(x^*)$ would violate the implication of Q that $f(z^*) \geq t^*f(x^*) + (1-t^*)f(y^*) > f(x^*)$. Hence we can conclude that $f(x^*) = f(y^*)$. Now $f(z^*) \leq f(x^*)$ implies $f(z^*) = f(x^*)$ so that we have three points x^*, y^*, and z^* such that $f(x^*) = f(y^*) = f(z^*)$. Concavity now implies that $f(\cdot) = f(x^*)$ on the entire interval $[x^*, y^*]$, i.e., the function is flat on this interval. Moreover, if there were to exist $z' \notin [x^*, y^*]$ such that $f(z') > f(x^*)$, then concavity would be violated. (Easy to prove, I won't bother.) Hence, $f(\cdot)$ must attain a global max on $[x^*, y^*]$ and thus satisfy S.

e) (¬P Q ¬R S ¬T)

Ans: No. ¬P and Q implies R. See the answer to the preceding part.

f) (¬P Q R ¬S ¬T)

Ans: Yes $f(x) = \begin{cases} x \text{ if } x < 0 \\ 0 \text{ if } x \geq 0 \end{cases}$.

g) (¬P Q R S T)

Ans: Yes. $f(x) = 0$.

h) (P ¬Q ¬R S ¬T)

Ans: No. P implies ¬S. To see this, we'll show that S implies ¬P. Assume that f is globally minimized at $\hat{x} \in \mathbb{R}$, and consider $\underline{x} < \hat{x} < \bar{x}$. By definition of $\hat{x}$, we have $f(\underline{x}) \geq f(\hat{x})$ and $f(\bar{x}) \geq f(\hat{x})$. In order for f to satisfy P, there must exist $\epsilon > 0$ such that $f(0.5\underline{x}+0.5\hat{x}) \geq f(\hat{x})+\epsilon$ and $f(0.5\bar{x} + 0.5\hat{x}) \geq f(\hat{x}) + \epsilon$. If this condition were satisfied, then both $(0.5\underline{x} + 0.5\hat{x})$ and $(0.5\bar{x} + 0.5\hat{x})$ would belong to the upper contour set corresponding to $f(\hat{x}) + \epsilon$. Now $\hat{x}$ belongs to the interval $(0.5\underline{x} + 0.5\hat{x}), (0.5\bar{x} + 0.5\hat{x})$ but does *not* belong to the upper contour set of f corresponding to $f(\hat{x}) + \epsilon$. Hence this set is not convex, establishing that f does not satisfy P.

i) (P ¬Q R ¬S T)

Ans: Yes. $f(x)$ is the pdf of a normal distribution.

j) (P Q R ¬S T)

Ans: No. ¬S and T implies ¬Q. To see this, pick $x_1, x_2 \in \mathbb{R}$ such that $y_2 = f(x_2) < y_1 = f(x_1)$. x_2 exists because $f(\cdot)$ is not minimized at x_1. Since f is bounded below, the image of f has a greatest lower bound, which we will denote by $\underline{y}$. Since $f(\cdot)$ is not minimized at x_2, $\underline{y} < y_2$. Therefore there exists $\lambda \in (0,1)$ such that $y_2 = \lambda y_1 + (1-\lambda)\underline{y}$. Now pick $x_3 = (x_2 - \lambda x_1)/(1-\lambda)$ and note that $x_2 = \lambda x_1 + (1-\lambda)x_3$. Let $y_3 = f(x_3)$; since $\underline{y}$ is the infimum of the image of f and since f does not attain a minimum, $y_3 > \underline{y}$. Now by construction, the point $(x_2, \lambda y_1 + (1-\lambda)y_3)$. lies on the line segment joining (x_1, y_1) and (x_3, y_3). But since $y_3 > \underline{y}$, $\lambda y_1 + (1-\lambda)y_3 > f(y_2)$. violating the condition for f to be concave, i.e., that the line segment joining any two points below the graph of f lies below the graph.

Problem 3: Metrics and Topology. (20 points)
The symbol $\mathbb{R}^n_{++}$ denotes the set of strictly positive vectors in $\mathbb{R}^n$.

a) The *max metric* on $\mathbb{R}^n$, written d^∞ is defined by, for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $d^\infty(\mathbf{x}, \mathbf{y}) = \max\{|x_i - y_i| : i = 1, ..., n\}$. Prove that d^∞ satisfies the triangle inequality. (**Help:** You may use the fact that the function $d^1 : \mathbb{R} \times \mathbb{R} \Rightarrow \mathbb{R}$, defined by $d^1(x, y) = |y - x|$ is a metric.)

Ans: Fix $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and select $j \in \{1, ..., n\}$ such that $d^\infty(\mathbf{x}, \mathbf{y}) = |x_j - y_j|$. Now pick $\mathbf{z} \in \mathbb{R}^n$ arbitrarily, and $j^x, j^y \in \{1, ..., n\}$ such that $d^\infty(\mathbf{x}, \mathbf{z}) = |x_{j^x} - z_{j^x}|$ and $d^\infty(\mathbf{z}, \mathbf{y}) = |z_{j^y} - y_{j^y}|$. We now have

$$\begin{aligned} d^\infty(\mathbf{x}, \mathbf{y}) &= |x_j - y_j| \\ &\leq |x_j - z_j| + |z_j - y_j| \\ &\leq |x_{j^x} - z_{j^x}| + |z_{j^y} - y_{j^y}| \\ &= d^\infty(\mathbf{x}, \mathbf{z}) + d^\infty(\mathbf{z}, \mathbf{y}) \end{aligned}$$

b) Let F denote the set of all *bounded* functions mapping $\mathbb{R}$ to $\mathbb{R}$ and define the function d^{sup} on $F \times F$ as follows: for $f, g \in F$,

$$d^{\mathsf{sup}}(f, g) = \sup\{|f(z) - g(z)| : z \in \mathbb{R}\}$$

Prove that d^{sup} satisfies the triangle inequality on $F \times F$.

Ans: For all $z \in \mathbb{R}$, we have

$$|f(z) - g(z)| \leq |f(z) - h(z)| + |h(z) - g(z)|$$

Since the right hand side is weakly greater than the left hand side, for all z, the same inequality must hold for the sups, i.e.,

$$\begin{aligned} d^{\mathsf{sup}}(f, g) &= \sup\{|f(z) - g(z)| : z \in \mathbb{R}\} \\ &\leq \sup\left\{\left(|f(z^f) - h(z^f)| + |h(z^g) - g(z^g)| : z \in \mathbb{R}\right)\right\} \end{aligned}$$

But the sup of a sum is necessarily no greater than the sum of the sups, i.e.,

$$\begin{aligned} d^{\mathsf{sup}}(f,g) &\leq \sup\left\{|f(z^f)-h(z^f)| : z\in\mathbb{R}\right\} + \{|\{h(z^g)-g(z^g)| : z\in\mathbb{R}\} \\ &= d^{\mathsf{sup}}(f,h) + d^{\mathsf{sup}}(h,g) \end{aligned}$$

The remainder of this question relates to the function $\rho : (\mathbb{R}_+ \times \mathbb{R}_{++}) \times (\mathbb{R}_+ \times \mathbb{R}_{++}) \Rightarrow \mathbb{R}$, defined as follows: for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2_+$, with $x_2, y_2 > 0$:

$$\rho(\mathbf{x},\mathbf{y}) = \sup\{|x_1\cos(z/x_2) - y_1\cos(z/y_2)| : z \in \mathbb{R}\}$$

c) Prove that ρ is a metric on $\mathbb{R}^2_{++}$.

Ans: To prove that ρ is a metric, we need to verify the four conditions given on p. 4 of the lecture note ANALYSIS1.

i) symmetry is immediate, from the absolute value operator.
ii) Nonnegativity is also immediate, again from the absolute value operator.
iii) The triangle inequality holds from the preceding part: for any $\mathbf{x} > 0$, $x_1\cos(\cdot/x_2)$ is bounded above by x_1 and below by $-x_1$, and it is clearly continuous, hence $x_1\cos(\cdot/x_2)$ belongs to F.
iv) Clearly, if $\mathbf{x} = \mathbf{y}$ then $\rho(\mathbf{x},\mathbf{y}) = 0$. Now suppose that $\mathbf{x} \neq \mathbf{y}$. We need to show that $\rho(\mathbf{x},\mathbf{y}) > 0$.
   (i) if $x_1\cos(0) \neq y_1\cos(0)$, then $\rho(\mathbf{x},\mathbf{y}) \geq |(x_1\cos(0) - y_1\cos(0)| > 0$.
   (ii) Suppose that $x_1\cos(0) = y_1\cos(0)$, so that $x_1 = y_1$. Since $\mathbf{x} \neq \mathbf{y}$ it must be the case that $x_2 \neq y_2$. Assume without loss of generality that $x_2 < y_2$ and set $z = x_2\pi/2$. We have

$$\begin{aligned} \rho(\mathbf{x},\mathbf{y}) &\geq |x_1\cos(x_2\pi/(2x_2)) - y_1\cos(x_2\pi/(2y_2))| \\ &= x_1\,|\cos(\pi/2) - \cos(x_2\pi/(2y_2))| \\ &= x_1\,|0 - \cos(x_2\pi/(2y_2))| \end{aligned}$$

Since $x_2/y_2 < 1$, and $\cos(\cdot)$ is positive on $(0,\pi/2)$, we know that $\cos(x_2\pi/(2y_2)) > 0$. Hence $\rho(\mathbf{x},\mathbf{y}) > 0$.

d) Prove that ρ is not a metric on $(\mathbb{R}_+ \times \mathbb{R}_{++})$.

Ans: When $x_1 = y_1 = 0$, $\rho(\mathbf{x},\mathbf{y}) = 0$, whether or not $x_2 = y_2$. So ρ fails the third condition for a metric. It is, however, a pseudometric.

For the remainder of this question, the universe is $\mathbb{R}^2_{++}$.

e) Fix $\mathbf{x} = (1,1)$ and $\mathbf{y} = (1,2)$.
   i) On the same axes, plot the graphs of $x_1\cos(\cdot/x_2)$ and $y_1\cos(\cdot/y_2)$ on the interval $[0, 4\pi]$.
   ii) From your graphs, what is $\rho(\mathbf{x},\mathbf{y})$?

Ans: $\rho(\mathbf{x}, \mathbf{y}) = 2$

A useful fact that is a little tricky to prove is:

$$\text{for all } \alpha, \gamma > 0, \text{ for all } \beta \neq \alpha, \rho((\gamma, \alpha), (\gamma, \beta)) \geq \gamma.$$

You can use this fact in the remaining parts of this question, without proving it.

f) Let $\mathbf{x} = (1, 1)$ and fix $\epsilon < 1$ and characterize (analytically, pictures optional) the set $B_\rho\left(\mathbf{x}, \epsilon | \mathbb{R}^2_{++}\right)$.

Ans: $\{(x_1, 1) : x_1 \in (1 - \epsilon, 1 + \epsilon)\}$

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g) For each of the following subsets of $\mathbb{R}^2_{++}$, specify whether the set is either
- open
- closed
- both
- neither

If your answer implies that a set is *not* open, then write down a boundary point of the set which is contained in the set; if your answer implies that a set is *not* closed, then write down an element of the closure of the set which does not belong to the set itself.

i) $\{1\} \times \mathbb{R}_{++}$ (i.e., $x_1 = 1$, $x_2 > 0$).

Ans: closed; the point $\mathbf{x} = (1, 1)$ is a boundary point, which is contained in the set

ii) $\mathbb{R}_{++} \times \{1\}$ (i.e., $x_1 > 0$, $x_2 = 1$).

Ans: both

iii) $[1, 2] \times [1, 2]$.

Ans: closed; the point $\mathbf{x} = (1, 1)$ is a boundary point, which is contained in the set

iv) $(1, 2) \times (1, 2)$.

Ans: open; the point $\mathbf{x} = (1, 1)$ belongs to the closure of the set but does not belong to the set

v) $[1, 2] \times (1, 2)$.

Ans: closed; the point $\mathbf{x} = (1, 1)$ is a boundary point, which is contained in the set

vi) $(1, 2) \times [1, 2]$.

Ans: open; the point $\mathbf{x} = (1, 1)$ belongs to the closure of the set but does not belong to the set

h) One of the following statements is true, the other is false. Prove the one that is true, and provide a counter-example (plus brief explanation) of the one that is false.

i) (a sequence $(\mathbf{x}^n) = (x_1^n, x_2^n)$ converges to $(\alpha, \beta) \in \mathbb{R}^2_{++}$ with respect to the metric ρ) implies ($\exists N \in \mathbb{N}$ such that $\forall n > N$, $x_1^n = \alpha$).

Ans: false; the sequence $(\alpha + 1/n, \beta)$ converges to (α, β), but for all n $x_1^n \neq \alpha$.

ii) (a sequence $(\mathbf{x}^n) = (x_1^n, x_2^n)$ converges to $\bar{x} = (\alpha, \beta) \in \mathbb{R}^2_{++}$ with respect to the metric ρ) implies ($\exists N \in \mathbb{N}$ such that $\forall n > N$, $x_2^n = \beta$).

Ans: true; The requirement for convergence is that $\forall \epsilon > 0$, $\exists N$, s.t. $\forall n > N$, $\rho(\mathbf{x}^n, \bar{x}) < \epsilon$. Now consider a sequence $(\mathbf{x}^n)$ with the property that $\forall N$, $\exists n > N$ such that $x_2^n \neq \beta$, In this case, $\rho(\mathbf{x}^n, \bar{x}) \geq \rho((\alpha, x_2^n), \bar{x}) \geq \alpha$.

Problem 4: Concavity, convexity, quasi-concavity, quasi-convexity. (20 points)

a) Consider the function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = \begin{cases} 0 \text{ if } x \leq 0 \\ 1 \text{ if } x > 0 \end{cases}$. Is this function concave? If so, prove it. If not, prove that this function is not concave.

Ans: f is not concave. To verify this, let $x = -1$ and $y = 0$. The pair $(-0.5, 0.5)$ is a convex combination of $(x, f(x)) = (-1, 0)$ and $(y, f(y)) = (0, 1)$, but lies above the pair $(z, f(z)) = (-0.5, 0)$. That is, $(-0.5, 0.5)$ does not belong to the hypograph of f. Hence the hypograph of f is not a convex set.

b) Can you construct a convex set $A \subset \mathbb{R}$ such that when the function f defined in the previous part is restricted to this domain, it is both discontinuous and concave? If so, provide an example of such an A and prove that f restricted to this domain is concave. (You don't need to actually *prove* that the function is discontinuous on A.) If not, prove that for any convex set A, if f restricted to A is discontinuous, then it cannot be concave.

Ans: Let A be the convex set $[0, 1]$. To prove that f restricted to this domain is concave, pick $x, y \in A$, $g(x) \leq f(x)$, and $g(y) \leq f(y)$. That is, the points $(x, g(x))$ and $(y, g(y))$ belong to the hypograph of f, restricted to A. We need to show that the line segment joining $(x, g(x))$ and $(y, g(y))$ is also contained in the hypograph. If x and y both belong to $(0, 1]$, this property is obvious. Assume therefore that $x = 0 < y$, so that $g(x) \leq 0$ and $g(y) \leq 1$. In this case, also the line segment joining $(x, g(x))$ and $(y, g(y))$ clearly lies below the graph of f.

Now let F denote the set of functions from $\mathbb{R}^n$ to $\mathbb{R}$.

c) Can a function in F be strictly-quasi-concave but not continuous? If so provide an example and *prove* that your example is strictly quasi-concave. If not, prove that strict quasi-concavity implies continuity; also provide a graphical example that illustrates your proof.

Ans: Yes: Example, $f(x) = \begin{cases} x \text{ if } x < 0 \\ 1 + x \text{ if } x \geq 0 \end{cases}$.

d) Can a function in F be quasi-concave but not continuous? If so provide an example. If so, provide an example and *prove* that your example is quasi-concave. If not, prove that quasi-concavity implies continuity; also provide a graphical example that illustrates your proof.

Ans: Yes. A strictly quasi-concave function is quasi-concave; in our answer to the previous part, we gave an example of a strictly quasi-concave function that is not continuous.

e) Give an example of a function in F that is strictly quasi-concave and strictly quasi-convex, but neither concave nor convex. You should give the formula for your function and then sketch it. Argue graphically that all the required properties are satisfied. (You don't need a formal proof, but you do need a convincing argument.)

Ans: $f(x) = x^3$.

f) For this part, let G denote the set of twice differentiable functions from $\mathbb{R}^1$ to $\mathbb{R}^1$ that are both strictly quasi-concave and strictly quasi-convex, but neither concave nor convex (i.e., not concave *and* not convex). Now consider the following conditions. (Reminder: the notation $g(\cdot) \geq 0$ means $\forall x, g(x) \geq 0$.)
A1: *either* $f'(\cdot) \geq 0$ *or* $f'(\cdot) \leq 0$.
A2: *either* $f'(\cdot) > 0$ *or* $f'(\cdot) < 0$.
A3: f is strictly monotonic.[1]
B1: $\exists x$ s.t. $f''(x) \geq 0$ *and* $\exists y$ s.t. $f''(y) \leq 0$
B2: $\exists x$ s.t. $f''(x) > 0$ *and* $\exists y$ s.t. $f''(y) < 0$
For each of the six combinations of "A" conditions and "B" conditions, (i.e., A1 & B1, A1 & B2, etc), state whether the pair of conditions is
i) *necessary but not sufficient* for $f \in G$ (prove necessary, provide counterexample to sufficient).
ii) *sufficient but not necessary* for $f \in G$ (prove sufficient, provide counterexample to necessary).
iii) *necessary and sufficient* for $f \in G$ (prove necessary and sufficient).
iv) *neither necessary nor sufficient* for $f \in G$ (provide counter examples to necessary and to sufficient).
To answer this question, you may use the following results.
T1: $f \in G$ is concave iff $\forall x$ $f''(x) \leq 0$.
T2: for $f \in G$ if $\forall x$ $f''(x) < 0$, then f is strictly concave.
T3: $f \in G$ is convex iff $-f$ is concave
T4: $f \in G$ is strictly convex iff $-f$ is strictly concave

Ans: The six combinations are:

A1,B1: these conditions are necessary but not sufficient. To see that they are necessary, suppose that B1 is violated. In this case *either* $f''(\cdot) > 0$ *or* $f''(\cdot) < 0$, in which case from T1 or T3, f is either (strictly) concave or (strictly) convex. To see that they are not sufficient, the function $f(x) = x$ satisfies both assumptions, yet is strictly quasi-concave, strictly quasi-convex, concave and convex.

A2,B1: these conditions are neither necessary nor sufficient. To see that they are not sufficient, $f(x) = x$ again satisfies both assumptions, yet is both concave and convex. To see that they are not necessary, the function $f(x) = x^3$ is strictly quasi-concave, strictly quasi-convex, not concave and not convex, yet violates A2.

[1] strictly monotonic means *either* strictly increasing *or* strictly decreasing.

A3,B1: these conditions are necessary but not sufficient. We have already shown that B1 is necessary. To see that they are not sufficient, consider again, $f(x) = x$.

A1,B2: these conditions are necessary but not sufficient. To see that they are necessary, suppose that B2 is violated. In this case *either* $f''(\cdot) \geq 0$ *or* $f''(\cdot) \leq 0$, in which case from T1 or T3, f is either concave or convex. To see that they are not sufficient, consider the function defined by $f(x) = \begin{cases} (x+1)^3 \text{ if } x \leq -1 \\ (x-1)^3 \text{ if } x \geq 1 \\ 0 \text{ if } x \in (-1,1) \end{cases}$ This function is not strictly quasi-concave, not strictly quasi-convex, not concave and not convex.

A2,B2: these conditions are sufficient but not necessary. To see that they sufficient, note that A2 implies A3; we show below that (A3,B2) is sufficient. To see that they are not necessary, consider again, $f(x) = x^3$, which, as we have seen, is strictly quasi-concave, strictly quasi-convex, not concave and not convex, but violates A2.

A3,B2: these conditions are necessary and sufficient. We have already shown that B2 is necessary. To see that they are sufficient, assume without loss of generality that f is monotone increasing. Consider the upper and lower contour sets associated with $\alpha \in \mathbb{R}$: if there exists $x \in \mathbb{R}$ such that $f(x) = \alpha$ then the lower (resp. upper) contour set corresponding to α is $(-\infty, x]$ (resp. $[x, \infty)$); otherwise the upper and lower contour sets are either $\emptyset$ or $\mathbb{R}$. In either case, these sets are convex. Moreover, since the function is strictly monotone, it has no flat spots. Therefore, the function is both strictly quasi-concave and strictly quasi-convex. B2 implies that neither f nor $-f$ satisfy T1. Hence by T1 and T3, the function is neither concave nor convex.

Problem 5: Necessary and sufficient conditions. (10 points)

Consider the function $f(x) = ax^3 + bx^2 + cx + d$, where $a, b, c, d \in \mathbb{R}$.

a) Give conditions on a, b, c, d that are necessary and sufficient for $f(\cdot)$ to obtain a global maximum on $\mathbb{R}$.

Ans:

$$a = 0,\ b \leq 0 \text{ and } (b = 0 \Rightarrow c = 0).$$

If these conditions are satisfied with $b \neq 0$ then f attains a max at $x = -c/2b$. If they are satisfied with $b = 0$ then f is constant. Now suppose that any of the conditions are violated:

i) if $a \neq 0$, then obviously the function increases or decreases without bound.
ii) if $b > 0$, the function increases without bound
iii) if $b = 0$ and $c \neq 0$ then the function increases or decreases without bound.

b) Give conditions on a, b, c, d that are necessary and sufficient for $f(\cdot)$ to obtain a unique global maximum on $\mathbb{R}$.

Ans:

$$a = 0,\ b < 0$$

c) Give conditions on a, b, c, d that are necessary and sufficient for $f(\cdot)$ to obtain a global maximum on $\mathbb{R}_+$.

Ans:

$$a \leq 0,\ (a = 0 \Rightarrow b \leq 0),\ (a = b = 0 \Rightarrow c < 0).$$

Problem 6: The Mantra. (20 points)

Consider the following nonlinear programming problem.

$$\begin{aligned} \text{maximize } f(\mathbf{x}) &= \cos(x_1 + 2x_2) s.t. \\ x_1 &\geq 0 \\ x_2 &\geq 0 \\ x_1 + x_2 &\geq \pi/4 \\ x_1 + x_2 &\leq 3\pi/4 \end{aligned}$$

The mantra (part 1) states that a necessary condition for $\mathbf{x}^*$ to solve this problem is that the gradient of f at $\mathbf{x}^*$ belongs to the nonnegative cone defined by the gradient vector(s) of the constraint(s) that are satisfied with equality at $\mathbf{x}^*$. (If there are no constraints satisfied with equality at $\mathbf{x}^*$, then the gradient of f at $\mathbf{x}^*$ must be zero.)

Hint: The gradient of f at $\mathbf{x} = (x_1, x_2)$ is $\begin{bmatrix} -1 \\ -2 \end{bmatrix} \sin(x_1 + 2x_2)$

a) Find all points in the constraint set at which the mantra (part 1) is satisfied.
Hint: We suggest you check the three distinct cases in the following order.
   i) the interior of the constraint set
   ii) the four line segments (excluding the corners)
   iii) four corners

Ans: The constraint set is drawn in Fig. 1. The boundaries of the set are indicated by heavy solid (green) lines. Following the hint, we consider each of the three cases in turn.

i) *the interior of the constraint set*: Since no constraints are satisfied with equality, the mantra will be satsified on the interior only if the gradient of f is zero. From the hint, this will be true iff $\sin(x_1 + 2x_2) = 0$, which in turn implies that $x_1 + 2x_2 = n\pi$, or $x_1 = n\pi - 2x_2$, for some $n \in \mathbb{N} \cup \{0\}$. Now the line $x_1 = n\pi - 2x_2$ intersects the constraint set only if $n = 1$. This line is represented in the figure by the light blue dashed curve.

ii) *the four line segments (excluding the corners)*; From the helpful hint, the gradient of f belongs to the vector space defined by the vector $(-1, -2)$. The gradients of the four constraints are, starting from the vertical line and proceeding clockwise, $(-1, 0)$, $(1, 1)$, $(0, -1)$ and $(-1, -1)$. Therefore, the mantra will be satisfied on the interior of one of the line segments iff $\sin(x_1 + 2x_2) = 0$, i.e., $x_1 = \pi - 2x_2$. This condition is satisfied at two points on the interiors of the four line segments, indicated on the figure by light blue dots.

iii) *four corners*: At each corner, two constraints are satisfied with equality. In Fig. 1, we indicate the gradients corresponding to each constraint with solid, green arrows and the gradient for the objective by a dotted, magenta arrow. Observe that the mantra is satisfied at two of

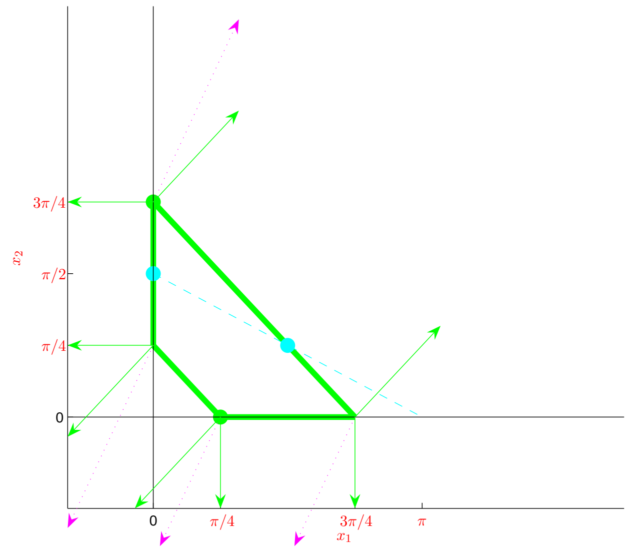

FIGURE 1. Graphical depiction of the mantra problem

the four corners—the points $(\pi/4, 0)$ and $(0, 3\pi/4)$. In the figure, these two corners are indicated by green dots.

b) Find the solution to this problem and the **x**-value(s) in the constraint set at which the maximum is attained.

Ans: The easiest way to find the maximum in this problem is to compute the value of the objective at each point where the mantra is satisfied. Along the light blue line, the objective is -1; at the point $(0, 3\pi/4)$, it is -0.7071. At $(\pi/4, 0)$, it is 0.7071. Hence the problem has a unique solution, at $(\pi/4, 0)$.

c) Which of the constraints are binding?

Ans: Since the gradient of the objective points into the positive cone defined by the gradients of the two constraints that are satisfied with equality at the solution, we know that both constraints are binding.

d) At each of the four corners of the constraint set, and at the solution(s), indicate on your graph the gradients of the objective and the constraints that are satisfied with equality.

Ans: See Fig. 1.