**Fall 2005** **ARE211**

MIDTERM EXAM

DUE DATE: MON OCT 31, '05, 2 P.M. (LEO'S MAILBOX IN 207 GIANNINI)

**There are five questions**

## Rules, regulations and advice

a) You must work on this exam by yourself. You are not allowed to talk to anybody about it or exchange notes (not even old class notes) with anybody during the exam period. You can use *your own* notes, the lecture notes, the answer keys to the problem sets, as well as books such as Simon & Blume.

b) Please write up the answers neatly; otherwise points may be subtracted because your answer is not legible.

c) Conciseness is a virtue of proofs. We will **definitely** take points off for excessively long proofs. It is just not fair to give somebody the same points for writing a proof covering three pages that other people did perfectly in four lines. (**None of the questions requires a proof that is long, so if your answer is getting really long and messy, you must be missing something**)

d) Some questions, or parts of questions might require a little trick which might be difficult to see. So, please work on the easy questions, and the easy parts of questions that have hard parts, first and save the difficult questions for the end. Each question has a "theme," so if you are having trouble with figuring out what the question is getting at, try a different part to see if you can pick up the theme.

e) If you believe a question needs clarification, or even worse, is incorrect (I hope this doesn't happen) send me an email. If I consider it appropriate to reply, I will reply to the whole class so everybody gets the same response. You might want to check your email periodically, just to see if there's any correspondence. **Please note well:**
   i) last year, more than half the email questions I received were undecipherable, and had to be returned for further clarification. This was a *major* pain. Hence please follow the following simple instructions for submitting questions.
   ii) **always** specify the **precise** part of the question to which you are referring.
   iii) your questions will (should) always relate to very fine details of the question. So if you think I've mis-worded a question (which, by the way, is unlikely, though possible) specify in a full sentence the re-wording you think is correct. Write mathematical symbols in words, i.e., if you want to write $x \in B(x, \delta) \subset X$, write "x in B(x,delta) subset X" and I'll know what you mean.

**Problem 1**: Local vs Global Conditions. (15 points)

a) Let $f : \mathbb{R}^n \to \mathbb{R}$ and suppose that there exists $\bar{\mathbf{x}} \in \mathbb{R}^n$ and an open set $U$ containing $\bar{\mathbf{x}}$ such that $f(\bar{\mathbf{x}}) \leq f(\mathbf{x})$, for all $\mathbf{x} \in U$. *Do not assume that $f$ is a differentiable function.*
   i) Identify a condition on $f$ that is sufficient to ensure that $f(\bar{\mathbf{x}}) < f(\mathbf{x})$, for all $\mathbf{x} \in \mathbb{R}^n$. Prove that the condition is sufficient.
   ii) Identify a second condition on $f$ that is sufficient to ensure that $f(\bar{\mathbf{x}}) < f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$ and is a *strictly weaker* condition than the first one you identified. Prove that the condition is sufficient.
   iii) Demonstrate with an example that the second condition is strictly weaker than the first.

**Problem 2**: Vector Spaces. (20 points)

Fix $n > 2$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function and fix $\mathbf{x} \in \mathbb{R}^n$. For $i = 1, ...n$, define the row vector $\psi^i$ by

$$\psi^i_j \quad = \quad \begin{cases} 1 & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}$$

where $\psi^i_j$ denotes the $j$'th element of the row vector $\psi^i$. Let $g = \nabla f(\mathbf{x}) \in \mathbb{R}^n$ and consider the set of vectors $V = \{\mathbf{v}^1, \mathbf{v}^2, ..., \mathbf{v}^n\} \subset \mathbb{R}^{n+1}$, defined by $\mathbf{v}^i = (\psi^i, g_i) \in \mathbb{R}^{n+1}$, for each $i$.

a) for an arbitrary vector $\boldsymbol{\alpha} \in \mathbb{R}^n$, write down, in the most economical possible form (i.e., using the fewest symbols you can), the linear combination of the elements of $V$ where the weight on the $i$'th element of $V$ is $\alpha_i$.
b) Write down the vector space $W$ which is *the span* of $V$. (Hint: the ideal answer to this question is in the following form: "$w \in \mathbb{R}^?$ belongs to $W$ if and only ???)".
c) What is the dimension of $W$? Formally support your answer.
d) Verify that $V$ is a basis for $W$.
e) Write down an different basis for $W$, and verify that it is indeed a basis.
f) Write down a minimal spanning set for $W$ that contains $n + 1$ elements. Verify that it is a minimal spanning set, and that it is not a basis.
g) Write down a two-dimensional subspace of $W$.
h) $W$ corresponds to a familiar object in multivariable calculus. What is this object? Support your answer by relating $W$ to the definition of the object you've identified.
i) Let $w$ be a weighted combination of the elements of $V$, with the property that the norm of the vector of weights is unity. The object $w$ corresponds to another familiar object in multivariable calculus. What is this object? Support your answer by relating $w$ to the definition of the object you've identified.

**Problem 3**: Calculus. (15 points)
After he gave away the Chocolate Factory to Charlie, Willie Wonka turned to calculus. Being an different kind of guy, he decided he would do calculus a little differently. For a function $g : \mathbb{R}^2 \to \mathbb{R}$ and each $\mathbf{x} \in \mathbb{R}^2$, Willy defined

- the *positive werivative* of $g$ at $\mathbf{x}$, denoted $g_+(\mathbf{x})$, as $\lim_{|k|\to\infty} \frac{\left(g(\mathbf{x}+(1,1)/k)-g(\mathbf{x})\right)}{\sqrt{2}/k}$
- the *negative werivative* of $g$ at $\mathbf{x}$, denoted $g_-(\mathbf{x})$, as $\lim_{|k|\to\infty} \frac{\left(g(\mathbf{x}+(1,-1)/k)-g(\mathbf{x})\right)}{\sqrt{2}/k}$
- the *gwadient* of $g$ as $\triangle g(\cdot) = (g_+(\cdot), g_-(\cdot))$.
- the *wifferential* of $g$ at $\mathbf{x}$ is the linear function $L^{g,\mathbf{x}}(\mathbf{dx}) = \triangle g(\mathbf{x}) \cdot \mathbf{dx}$.

a) Using the wifferential, write an expression for the partial derivatives of $g$.

For the remainder of the question, let $f = xy^2$.

b) Write down the expression for $\triangle f(\cdot)$.
c) Write down the wifferential of $f$ at $(2,3)$
d) Using the wifferential, compute the partial derivatives of $f$.
e) Comment on the relationship between the wifferential of $f$ at $(2,3)$ and the differential of $f$ at $(2,3)$. (Hint: the ideal answer to this question includes a word starting with "v" and another starting with "b".)

**Problem 4**: Taylor Theory. (25 points)

Consider the CES production function $f(x,y) = x^\rho + y^\rho$, where $\rho \in (0,1)$. (Actually this is a transformation of a CES production function, but never mind.)

a) Write down the gradient and the Hessian of this function
b) Write down the second order Taylor expansion of $f$ at $(x,y;\rho)$. *Do not use matrix notation, i.e., multiply out the matrix.* Factor out as many terms as possible.
c) Now let $y = x$,
   i) express in the simplest possible way the expression for the second order Taylor expansion of $f(\cdot,\cdot;\rho)$ at $(x,x)$.
   ii) Characterize the conditions on $(dx,dy)$ under which the first order Taylor expansion has the same sign as the second order expansion? (Hint: there are two cases to consider). (Hint2: I really meant $(dx,dy)$ not $(dx,dx)$).
   iii) Fix $x,\rho$ and an element $\mathbf{v}$ of the unit circle. Let $\bar{\lambda}(x,\mathbf{v},\rho)$ denote the largest $\lambda$ such that for $\theta < \lambda$ the first order Taylor expansion, i.e., $\nabla f(x,x) \cdot \theta\mathbf{v}$ has the same sign as $(f(x+\theta v_1, x+\theta v_2;\rho) - f(x,x;\rho))$. Discuss the comparative statics of $\bar{\lambda}(\cdot,\cdot,\cdot)$ with respect to $\mathbf{x}$, $\mathbf{v}$ and $\rho$.
   iv) for two values of $\rho$, preferably 1/3 and 2/3, sketch the level sets of $f(\cdot,\cdot;\rho)$ that pass through (1,1) and (2,2). Illustrate diagramatically the comparative statics properties you've identified in the previous sub-question.

**Problem 5**: Second Order Conditions. (25 points)

a) For this part of the question, assume that $f$ is twice continuously differentiable and that the gradient of $f$ is never zero.

Let $\mathbb{T}(f, \mathbf{x})$ denote the plane that is tangent to the level set of $f$ corresponding to $f(\mathbf{x})$ (that is, $\mathbb{T}(f, \mathbf{x})$ is the set of points that are perpendicular to $\nabla f(\mathbf{x})$). Then $f$ is strictly quasi-concave if and only if for every $\mathbf{x}$, the set $\mathbb{T}(f, \mathbf{x}) \sim \{\mathbf{x}\}$ belongs to the strict lower contour set of $f$ corresponding to $f(\mathbf{x})$. (The set $A \sim \{\mathbf{x}\}$ consists of all the elements of $A$ excluding the element $\mathbf{x}$.)

Now suppose that $f$ satisfies the following condition:

$$\text{for all } \mathbf{x} \text{ and all } \mathbf{dx} \text{ such that } \nabla f(\mathbf{x})'\mathbf{dx} = 0, \mathbf{dx}'\mathrm{Hf}(\mathbf{x})\mathbf{dx} < 0 \tag{A}$$

i) Use one of Taylor's theorems to prove that if $f$ satisfies condition $A$ then it satisfies the above definition of strict quasi-concavity.

ii) Provide an example to establish that condition (A) is not necessary for the above definition of strict quasi-concavity to hold.

b) Consider the problem, maximize $f(\mathbf{x})$ s.t $g(\mathbf{x}) \leq b$, where $f$ and $g$ both map $\mathbb{R}^n$ to $\mathbb{R}$. Assume that $f$ and $g$ are both *concave* functions. In this question, we explore conditions on $g$ which ensure the following property

$$\text{if } \mathbf{x} \text{ satisfies the Kuhn Tucker conditions, then } \mathbf{x} \text{ solves the max problem.} \tag{S}$$

i) Show graphically that if $g$ is everywhere *less concave* than $f$, then property (S) holds. (Hint: concave functions are quasi-concave).

ii) Use one of Taylor's theorems to show that the following mathematical condition does *not* capture the notion of "less concave." Specifically, demonstrate that the condition (B) below *does not* imply that condition (S) is satisfied:

$$\text{for all } \mathbf{x} \text{ and all } \mathbf{dx}, \mathbf{dx}'\mathrm{Hf}(\mathbf{x})\mathbf{dx} < \mathbf{dx}'\mathrm{Hg}(\mathbf{x})\mathbf{dx} < 0 \tag{B}$$

(Hint: let $\nabla f(\mathbf{x}) = \begin{bmatrix} 1 & 1 \end{bmatrix}$, $\mathrm{Hf}(\mathbf{x}) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$, $\mathrm{Hg}(\mathbf{x}) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. Now all you have to do is find $\nabla g(\mathbf{x})$ and pick a vector $\mathbf{dx}$ such that one of the second order Taylor expansions is positive, and the other is negative. Of course, you can't just stop there: you have to explain why these properties answer the question.)

iii) Modify condition (B) so that it *does* capture the notion of "less concave". (Hint: your condition should exhibit the property that whenever one of the second order Taylor expansions is positive then the other one is also. Of course, you can't just stop there: you have to explain why these properties answer the question.)