DUE DATE: MON OCT 25, '04, 11 A.M. (LEO'S MAILBOX IN 207 GIANNINI)

There are four questions, each is worth 25 points.

Rules, regulations and advice

a) You must work on this exam by yourself. You are not allowed to talk to anybody about it or exchange notes (not even old class notes) with anybody during the exam period. You can use *your own* notes, the lecture notes, the answer keys to the problem sets, as well as books such as Simon & Blume.

b) Please write up the answers neatly; otherwise points may be subtracted because your answer is not legible.

c) Conciseness is a virtue of proofs. We will **definitely** take points off for excessively long proofs. It is just not fair to give somebody the same points for writing a proof covering three pages that other people did perfectly in four lines. (**None of the questions requires a proof that is long, so if your answer is getting really long and messy, you must be missing something**)

d) Some questions, or parts of questions might require a little trick which might be difficult to see. So, please work on the easy questions, and the easy parts of questions that have hard parts, first and save the difficult questions for the end. Each question has a "theme," so if you are having trouble with figuring out what the question is getting at, try a different part to see if you can pick up the theme.

e) If some part of a question requires a metric, unless we specifically indicate otherwise, assume for that part of the question that it is the Pythagorian metric. **Any email of the form: "which metric do you mean?" will be instantly destroyed.**

f) If you believe a question needs clarification, or even worse, is incorrect (I hope this doesn't happen) send me an email. If I consider it appropriate to reply (cf (5)), I will reply to the whole class so everybody gets the same response. You might want to check your email periodically, just to see if there's any correspondence. **Please note well:**
   i) last year, more than half the email questions I received were undecipherable, and had to be returned for further clarification. This was a *major* pain. Hence please follow the following simple instructions for submitting questions.
   ii) **always** specify the **precise** part of the question to which you are referring.
   iii) your questions will (should) always relate to very fine details of the question. So if you think I've mis-worded a question (which, by the way, is unlikely, though possible) specify in a full sentence the re-wording you think is correct. Write mathematical symbols in words, i.e., if you want to write $x \in B(x,\delta) \subset X$, write "x in B(x,delta) subset X" and I'll know what you mean.

Problem 1 (25 points).

For this question, rule 5 does not apply.

Definition: Two metrics are *equivalent* if they define the same open sets, that is if a set is open with respect two the first metric whenever it is open with respect to the second.

The next definition applies *only* to part c) of this question.
Definition: Two metrics are *uniformly equivalent* given $\varepsilon > 0$, there exists $\delta > 0$ such that for all $\mathbf{x},\mathbf{y} \in X$,

$$\begin{aligned} \rho(\mathbf{x},\mathbf{y}) < \delta &\implies \sigma(\mathbf{x},\mathbf{y}) < \varepsilon \qquad \text{and} \\ \sigma(\mathbf{x},\mathbf{y}) < \delta &\implies \rho(\mathbf{x},\mathbf{y}) < \varepsilon \end{aligned}$$

a) Show that two metrics σ and ρ on a set X are equivalent if and only if given $\mathbf{x} \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for all $\mathbf{y} \in X$,

$$\rho(\mathbf{x},\mathbf{y}) < \delta \implies \sigma(\mathbf{x},\mathbf{y}) < \varepsilon \tag{1}$$
$$\sigma(\mathbf{x},\mathbf{y}) < \delta \implies \rho(\mathbf{x},\mathbf{y}) < \varepsilon \tag{2}$$

b) Show that the Pythagorian metric on $\mathbb{R}^n$ is equivalent to the metric ρ, defined by

$$\rho(\mathbf{x},\mathbf{y}) = \max\{|x_i - y_i| : i = 1,\dots,n\}$$

c) Prove that the following metrics on $\mathbb{R}_{++}$ are equivalent but not uniformly equivalent.

$$\begin{aligned} \rho(x,y) &= |x-y| \\ \sigma(x,y) &= |1/x - 1/y| \end{aligned}$$

Hint: you can identify certain upper bounds and assume without loss of generality that $\varepsilon > 0$ is not greater than these bounds.

Problem 2 (25 points).

a) Given $N \in \mathbb{N}$, $N > 2$, we say that a nonempty set W is an N-wector space if $\{\mathbf{v}^1,\dots,\mathbf{v}^N\} \subset W$ and $\alpha \in \mathbb{R}^N$ implies $\sum_{i=1}^N \alpha_i \mathbf{v}^i \in W$. Show that for any $N \in \mathbb{N}$, a set W is an N-wector space iff it is a vector space.
b) The remaining parts of this question relate to the following construction. Fix $\boldsymbol{\theta} \in \mathbb{R}^5$ and a set $K \subset \mathbb{N}$. Let

$$X(\boldsymbol{\theta},K) = \left\{ \text{sequences in } \mathbb{R}^5 \text{ s.t. } \begin{cases} x_n = \boldsymbol{\theta} & \text{for all } n \in K \\ x_{n,2} = x_{n,3} & \text{for all } n \in K^C \end{cases} \right\}$$

where $x_{n,j}$ denotes the $j'th$ component of the n'th element of the sequence. What is the *largest* collection of $\boldsymbol{\theta}$'s in $\mathbb{R}^5$ and largest collection of sets K's for which $X(\boldsymbol{\theta},K)$ is a finite dimensional vector space. To get full marks for this question, you must prove that for the pair of collections that you have identified,

i) whenever $(\boldsymbol{\theta},K)$ belongs to this pair of collections, then $X(\boldsymbol{\theta},K)$ is a finite dimensional vector space,

ii) whenever $(\boldsymbol{\theta}, K)$ does not belong to this pair of collections, then $X(\boldsymbol{\theta}, K)$ is not a finite dimensional vector space,

c) Fix a set $K \subset \mathbb{N}$ and $\boldsymbol{\theta} \in \mathbb{R}^5$ such that $X(\boldsymbol{\theta}, K)$ is a finite-dimensional vector space. Find a basis for $X(\boldsymbol{\theta}, K)$. Do this abstractly, not for a specific K and $\boldsymbol{\theta}$. That is, you should give one answer that "works" for all K and all $\boldsymbol{\theta}$ such that $X(\boldsymbol{\theta}, K)$ is a vector space. Demonstrate that it is a basis. Hint: it is quite possible that you have already partially or fully completed part c) in your answer to part b). If you have, simply refer to your previous answer; don't repeat work you've already done.

d) Given a set $K \subset \mathbb{N}$ and $\boldsymbol{\theta} \in \mathbb{R}^5$ such that $X(\boldsymbol{\theta}, K)$ is a finite dimensional vector space, what is the dimension of $X(\boldsymbol{\theta}, K)$?

e) Given a set $K \subset \mathbb{N}$ and $\boldsymbol{\theta} \in \mathbb{R}^5$ such that $X(\boldsymbol{\theta}, K)$ is a finite dimensional vector space, find a minimal spanning set for $X(\boldsymbol{\theta}, K)$ *that is not a basis*. Again, do this abstractly. Demonstrate that it spans, is minimal, but that it isn't a basis.

Problem 3 (25 points).

a) Let U denote the set of *all* 2×2 matrices for which no real eigenvalues exist. Is U open, closed or neither? Whatever your answer, specify the appropriate universe. Prove your answer.

Now for the remainder of this question, fix $\beta \in \mathbb{R}_+$ and $\alpha \in \mathbb{R}$ and let $A(\alpha|\beta) = \begin{bmatrix} 2 & \beta \\ \alpha & 4 \end{bmatrix}$. (I'd strongly recommend—who would have thought—that you use a computer to check your answers. If you want to use matlab, the following might help: `A = [ a , b ; c , d ]` will define a 2×2 matrix. `[Vec,Val] = eig(A)` will deliver its eigenvectors and eigenvalues. `help eig` will give you more details.)

b) Write down an expression (in terms of α and β) for the eigenvalues of $A(\alpha|\beta)$.

c) Compute the largest interval I in $\mathbb{R}$ such that $A(\cdot|\beta)$ has real eigenvalues on this interval.

d) For $\alpha \in I$, write down an expression for two distinct unit eigenvectors of $A(\alpha|\beta)$.

e) Let $\mathbf{v}^1(\alpha)$ and $\mathbf{v}^2(\alpha)$ denote the expressions for the unit eigenvectors you have just calculated and let $\mathrm{Cos}(\alpha)$ denote the cosine of the angle between them. Write down an expression for $\mathrm{Cos}(\alpha)$. (You will be surprised at how simple the expression is.)

f) Set $\beta = 2$ and (using a computer if you like) sketch a plot of $\mathrm{Cos}(\cdot)$ as a function of α. Interpret your graph in terms of the relationship between the two eigenvectors.

g) Based only on the data you have computed for this question,

   i) conjecture a necessary and sufficient condition for a 2×2 matrix to have pairwise orthogonal eigenvectors. For what class of matrix can you *prove* this conjecture, based *only* on results obtained by answering this question?

   ii) conjecture one property for the eigenvalues, and one property for the eigenvectors, of a 2×2 matrix which belongs to the boundary of the set of all 2×2 matrices that have real eigenvalues. For what class of matrices do you havea enough information to prove this answer?

   iii) As a consequence of the answer to part g)ii), the relationship between eigenvectors and non-eigenvectors is fundamentally different for matrices on the above boundary vs matrices in the interior of the set of matrices with real eigenvectors. Explain.

h) Construct 2 distinct 2×2 matrices with the property that (a) their eigenvalues and eigenvectors are identical; (b) the images of the unit circle for the two matrices are different? Hint: look at your answer to part g).

Problem 4 (25 points).

Let A be a symmetric $n \times n$ matrix with nonzero eigenvalues $(\lambda_1, \ldots, \lambda_n)$ and consider the difference equation system $\mathbf{x}^t = A\mathbf{x}^{t-1}$, for $t \in \mathbb{N}$. A *solution sequence* for this system is any sequence $(\mathbf{x}^t)_{t=1}^{\infty}$ such that for each t, $\mathbf{x}^t = A\mathbf{x}^{t-1}$. A *steady state* for this system is a solution sequence with the property that all elements of the sequence are equal.

a) Prove the following theorem: a necessary and sufficient condition for the zero sequence to be the *unique* steady state for the system defined by the matrix A is that none of A's eigenvalues is equal to unity.
b) Show that if $|\lambda_i| < 1$, for $i = 1, \ldots, n$, then for any solution sequence, $(\mathbf{x}^t)_{t=1}^{\infty}$, to the system, $\lim_{t\to\infty} \mathbf{x}^t = 0$.
c) Now suppose that $|\lambda_1| > |\lambda_i|$, $i = 2, \ldots, n$, and $|\lambda_1| > 1$. Let $\mathbf{v}^1$ be an eigenvector with eigenvalue λ_1. Let $(\mathbf{x}^t)_{t=1}^{\infty}$ be a solution sequence for the system. Characterize the limit behavior of this sequence in terms of the angle θ^t between $\mathbf{x}^t$ and $\mathbf{v}^1$. Your answer should be of the form: $\forall \varepsilon > 0$, there exists T such that if t and $\mathbf{x}^1$ satisfy certain conditions, then some property can be established about this angle. Prove your answer. (Hint: you need to identify a number of different cases. The number is bigger than 2, but smaller than 357.)