Problem 1 (25 points).

Definition: Two metrics are *equivalent* if they define the same open sets, that is if a set is open with respect two the first metric whenever it is open with respect to the second.

The next definition applies *only* to part c) of this question.
Definition: Two metrics are *uniformly equivalent* given $\varepsilon > 0$, there exists $\delta > 0$ such that for all $\mathbf{x}, \mathbf{y} \in X$,

$$\rho(\mathbf{x},\mathbf{y}) < \delta \quad \Longrightarrow \quad \sigma(\mathbf{x},\mathbf{y}) < \varepsilon \qquad \text{and}$$
$$\sigma(\mathbf{x},\mathbf{y}) < \delta \quad \Longrightarrow \quad \rho(\mathbf{x},\mathbf{y}) < \varepsilon$$

a) Show that two metrics σ and ρ on a set X are equivalent if and only if given $\mathbf{x} \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for all $\mathbf{y} \in X$,

$$\rho(\mathbf{x},\mathbf{y}) < \delta \quad \Longrightarrow \quad \sigma(\mathbf{x},\mathbf{y}) < \varepsilon \tag{1}$$
$$\sigma(\mathbf{x},\mathbf{y}) < \delta \quad \Longrightarrow \quad \rho(\mathbf{x},\mathbf{y}) < \varepsilon \tag{2}$$

We'll first show that if either (1) or (2) fails, then we can construct an open set with respect to one metric that is not open with respect to the other. Assume that (1) fails, i.e., there exists $\mathbf{x} \in X$, $\varepsilon > 0$ and a sequence (y_n) such that for each n, $\rho(\mathbf{x}, y_n) < 1/n$ but $\sigma(\mathbf{x}, y_n) \geq \varepsilon$. Let $U = B_\sigma(\mathbf{x}, \varepsilon)$. Necessarily U is open w.r.t. σ and contains $\mathbf{x}$. However, the sequence (y_n) converges to $\mathbf{x}$ w.r.t. ρ, but none of the y_n belong to U. Hence $\mathbf{x}$ is a boundary point of U w.r.t. ρ and so cannot be open. A parallel argument can be constructed if (2) fails.

Now suppose that both (1) or (2) are satisfied. We need to show that a set U is open w.r.t. ρ iff it is open w.r.t. σ. We will do so by picking an arbitrary set U that is open w.r.t. σ and showing that an point $\mathbf{x} \in U$ is an interior point of U w.r.t. ρ. This will show that every element of U is an interior point w.r.t. ρ, and thus that U is open w.r.t. ρ. Since U is open w.r.t. σ, there exists $\varepsilon > 0$ such that $B_\sigma(\mathbf{x}, \varepsilon) \subset U$. From (1), there exists $\delta > 0$, such that $B_\rho(\mathbf{x}, \delta) \subset B_\sigma(\mathbf{x}, \varepsilon) \subset U$. That is $\mathbf{x}$ belongs to a ρ-open subset of U, and is hence a ρ-interior point of U. A parallel argument using (1) can be constructed to show that if U is open w.r.t. ρ, then it is also open w.r.t. σ.

b) Show that the Pythagorian metric on $\mathbb{R}^n$ is equivalent to the metric ρ, defined by

$$\rho(\mathbf{x},\mathbf{y}) \quad = \quad \max\{|x_i - y_i| : i = 1, \ldots, n\}$$

We'll refer to the Pythag metric as σ. Fix $\mathbf{x} \in \mathbb{R}^n$ and $\varepsilon > 0$.and let $\delta = \varepsilon/n$. It's easier to prove that

$$\sigma(\mathbf{x},\mathbf{y}) \geq \varepsilon \quad \Longrightarrow \quad \rho(\mathbf{x},\mathbf{y}) \geq \delta \tag{3}$$
$$\rho(\mathbf{x},\mathbf{y}) \geq \varepsilon \quad \Longrightarrow \quad \sigma(\mathbf{x},\mathbf{y}) \geq \delta \tag{4}$$

Clearly, if $\rho(\mathbf{x},\mathbf{y}) \geq \varepsilon$ then $\sigma(\mathbf{x},\mathbf{y}) = \sqrt{\varepsilon^2 + K}$, for some nonnegative number K. Hence $\sigma(\mathbf{x},\mathbf{y}) \geq \varepsilon$. On the other hand, if $\sigma(\mathbf{x},\mathbf{y}) \geq \varepsilon$ then necessarily $|x_i - y_i| \geq \varepsilon/\sqrt{n} > \varepsilon/n$, for at least one i. But this implies that $\rho(\mathbf{x},\mathbf{y}) > \delta$.

c) Prove that the following metrics on $\mathbb{R}_{++}$ are equivalent but not uniformly equivalent.

$$\begin{aligned}\rho(x,y) &= |x-y| \\ \sigma(x,y) &= |1/x-1/y|\end{aligned}$$

Hint: you can identify certain upper bounds and assume without loss of generality that $\varepsilon > 0$ is not greater than these bounds.

To show that these are equivalent, pick $x \in \mathbb{R}$ and $\varepsilon > 0$. Assume without loss of generality that $\varepsilon < \min(0.5, 1/x)$. We'll set $\delta = \begin{cases} \min[\varepsilon/4, \varepsilon x(x-\varepsilon)] & \text{if } x < 1.5 \\ \varepsilon/x(x+\varepsilon) & \text{if } x \geq 1.5 \end{cases}$.
Consider $y \in B_\sigma(x,\delta)$. We have two cases to consider:

i) $\boldsymbol{x < 1.5}$: In this case, $y < 2$ since if $y \geq 2$, then

$$\sigma(x,y) \geq |1/1.5-1/2| = 1/6 > 1/8 > \varepsilon/4$$

Therefore we have

$$\sigma(x,y) = |x-y|/xy < \delta \leq \varepsilon/4$$

Since $xy < 4$, $|x-y| = \rho(x,y) < \varepsilon$.

ii) $\boldsymbol{x \geq 1.5}$: Note first that $y < x+\varepsilon$, since if $y \geq x+\varepsilon$, then

$$\sigma(x,y) = |1/x-1/y| \geq |1/x-1/(x+\varepsilon)| = \varepsilon/x(x+\varepsilon) = \delta$$

Therefore we have

$$\sigma(x,y) = |1/x-1/y| = |x-y|/xy < \delta = \varepsilon/x(x+\varepsilon)$$

so that

$$\rho(x,y) = |x-y| < \delta xy = \varepsilon xy/x(x+\varepsilon) < \varepsilon$$

Now consider $y \in B_\rho(x,\delta)$. We again have two cases to consider:

i) $\boldsymbol{x < 1.5}$: Since $y > x-\delta > x-\varepsilon$, we have

$$\sigma(x,y) = |x-y|/xy < |x-y|/x(x-\varepsilon) < \delta/x(x-\varepsilon) \leq \varepsilon x(x-\varepsilon)/x(x-\varepsilon) = \varepsilon$$

ii) $\boldsymbol{x \geq 1.5}$: Since $y > x-\delta > x-\varepsilon > 1.5-0.5 = 1$, we have

$$\sigma(x,y) = |x-y|/xy < |x-y| < \delta < \varepsilon$$

However, the two metrics are not uniformly equivalent. To see this set $\varepsilon = 1$ and for all $n \in \mathbb{N}$, let $\delta_n = 1/n$. Let $x = 1/n$ and $y = 1/2n$. For all n, $\rho(x,y) = |1/n-1/2n| = 1/2n < \delta_n$. However, $\sigma(x,y) = |2n-n| \geq 1 = \varepsilon$.

Problem 2 (25 points).

a) Given $N \in \mathbb{N}$, $N > 2$, we say that a nonempty set W is an N-wector space if $\{\mathbf{v}^1, \ldots, \mathbf{v}^N\} \subset W$ and $\alpha \in \mathbb{R}^N$ implies $\sum_{i=1}^N \alpha_i \mathbf{v}^i \in W$. Show that for any $N \in \mathbb{N}$, a set W is an N-wector space iff it is a vector space.

If $N \leq 2$, there's nothing to prove, so assume $N > 2$. The proof in one direction is completely trivial. Suppose W is an N-wector space. Now consider $\{\mathbf{v}^1, \mathbf{v}^2\} \subset W$ and $\alpha \in \mathbb{R}^2$. Extend α to $\mathbb{R}^n$ by adding zeros and let $\{\mathbf{v}^3, \ldots, \mathbf{v}^N\}$ be arbitrarily chosen. We have $\sum_{i=1}^2 \alpha_i \mathbf{v}^i = \sum_{i=1}^N \alpha_i \mathbf{v}^i \in W$, proving that W is a vector space.

Now suppose W is an vector space. Trivially, we know that W is also a 2-wector space. Now assume that for some $n \geq 2$, we've proved that W is an n-wector space. (We have done so for $n = 2$.) We'll prove that W is also an $(n+1)$-wector space. Arbitrarily pick $\{\mathbf{v}^1, \ldots, \mathbf{v}^{n+1}\} \subset W$ and $\alpha \in \mathbb{R}^{n+1}$. Let $\mathbf{w} = \sum_{i=1}^n \alpha_i \mathbf{v}^i$ and note that by assumption $\mathbf{w} \in W$. Since W is a vector space $\sum_{i=1}^{n+1} \alpha_i \mathbf{v}^i = \mathbf{w} + \alpha_{n+1} \mathbf{v}^{n+1} \in W$. Therefore, W is an $(n+1)$-wector space.

b) The remaining parts of this question relate to the following construction. Fix $\boldsymbol{\theta} \in \mathbb{R}^5$ and a set $K \subset \mathbb{N}$. Let

$$X(\boldsymbol{\theta}, K) = \left\{ \text{sequences in } \mathbb{R}^5 \text{ s.t. } \begin{cases} x_n = \boldsymbol{\theta} & \text{for all } n \in K \\ x_{n,2} = x_{n,3} & \text{for all } n \in K^C \end{cases} \right\}$$

where $x_{n,j}$ denotes the $j'th$ component of the n'th element of the sequence. What is the *largest* collection of $\boldsymbol{\theta}$'s in $\mathbb{R}^5$ and largest collection of sets K's for which $X(\boldsymbol{\theta}, K)$ is a finite dimensional vector space. To get full marks for this question, you must prove that for the pair of collections that you have identified,

i) whenever $(\boldsymbol{\theta}, K)$ belongs to this pair of collections, then $X(\boldsymbol{\theta}, K)$ is a finite dimensional vector space,

ii) whenever $(\boldsymbol{\theta}, K)$ does not belong to this pair of collections, then $X(\boldsymbol{\theta}, K)$ is not a finite dimensional vector space,

Let Θ consist of all *co-finite* subsets of $\mathbb{N}$. A subset of $\mathbb{N}$ is co-finite if its *complement* in $\mathbb{N}$, denoted K^C, is a finite set. Let $\boldsymbol{\theta} = \mathbf{0}$. Pick an arbitrary co-finite subset $K \subset \mathbb{N}$, pick sequences $x, y \in X(\mathbf{0}, K)$, $\alpha, \beta \in \mathbb{R}$ and let z denote the sequence $\alpha x + \beta y$. For all $n \in K$, $z_n = \alpha x_n + \beta y_n = \alpha \mathbf{0} + \beta \mathbf{0} = \mathbf{0}$, while for all $n \in K^C$, $z_{n,2} = z_{n,3}$. Therefore, $z \in X(\mathbf{0}, K)$, establishing that $X(\mathbf{0}, K)$ is a vector space.
The *only* way to prove that it's finite dimensional is to provide a basis for the space. For $k \in K^C$ and $i = 1, 2, 4, 5$, let $y^{k,i}$ denote the sequence defined by, for $n \in \mathbb{N}$ and $j = 1, \ldots, 5$

$$y^{k,i}_{n,j} = \begin{cases} 1 & \text{if n = k \& i = 2 \& j = 2,3} \\ 1 & \text{if n = k \& i} \neq \text{2 \& j = i} \\ 0 & \text{otherwise} \end{cases}.$$

We will establish that this is a basis in the answer to the next part. For now note simply that the number of sequences we have defined is 4 times the number of elements in K^C which is a finite number. Hence $X(\mathbf{0}, K)$ is finite dimensional.
On the other hand,

i) for $\boldsymbol{\theta} \neq \mathbf{0}$, pick $x, y \in X(\boldsymbol{\theta}, K)$ and let z denote the sequence $x+y$. For all $n \in K$, $z_n = x_n + y_n = \boldsymbol{\theta} + \boldsymbol{\theta} = 2\boldsymbol{\theta} \neq \boldsymbol{\theta}$. Therefore, $z \notin X(\boldsymbol{\theta}, K)$, so that $X(\boldsymbol{\theta}, K)$ is not a vector space.

ii) If K is not co-finite, then the number of elements in a basis for $X(\boldsymbol{\theta}, K)$ is infinite, so that $X(\boldsymbol{\theta}, K)$ is not finite dimensional.

c) Fix a set $K \subset \mathbb{N}$ and $\boldsymbol{\theta} \in \mathbb{R}^5$ such that $X(\boldsymbol{\theta}, K)$ is a finite-dimensional vector space. Find a basis for $X(\boldsymbol{\theta}, K)$. Do this abstractly, not for a specific K and $\boldsymbol{\theta}$. That is, you should give one answer that "works" for all K and all $\boldsymbol{\theta}$ such that $X(\boldsymbol{\theta}, K)$ is a vector space. Demonstrate that it is a basis. Hint: it is quite possible that you have already partially or fully completed part c) in your answer to part b). If you have, simply refer to your previous answer; don't repeat work you've already done.

A basis was provided in the answer to the previous part. Call it Y We now just have to check that it is indeed a basis. To verify this, we need to check that: (a) Y is a subset of $X(\mathbf{0}, K)$; (b) Y spans $X(\mathbf{0}, K)$; (c) any proper subset of Y will not span $X(\mathbf{0}, K)$. Clearly, each element of Y belongs to $X(\mathbf{0}, K)$. Moreover for an arbitrarily chosen $x \in X(\mathbf{0}, K)$, it is clearly the case that $x = \sum_{j=1,2,4,5} \sum_{n \in K^C} x_{n,j} y^{n,j}$. Hence Y spans $X(\mathbf{0}, K)$. Finally, suppose that $y^{k,i}$ were omitted from Y, for some $k \in K^C$ and $i = 1,2,4,5$. Since for all remaining $y \in Y$, $y_{k,i} = 0$, $y^{k,i}$ cannot be written as a linear combination of the remaining elements of Y. Hence Y is a set of basis vectors for $X(\mathbf{0}, K)$,

d) Given a set $K \subset \mathbb{N}$ and $\boldsymbol{\theta} \in \mathbb{R}^5$ such that $X(\boldsymbol{\theta}, K)$ is a finite dimensional vector space, what is the dimension of $X(\boldsymbol{\theta}, K)$?

The dimension of $X(\boldsymbol{\theta}, K)$ is the number of elements of any basis for $X(\boldsymbol{\theta}, K)$. As noted already, the dimension of $X(\boldsymbol{\theta}, K)$ is $4 \times \#K^C$.

e) Given a set $K \subset \mathbb{N}$ and $\boldsymbol{\theta} \in \mathbb{R}^5$ such that $X(\boldsymbol{\theta}, K)$ is a finite dimensional vector space, find a minimal spanning set for $X(\boldsymbol{\theta}, K)$ *that is not a basis*. Again, do this abstractly. Demonstrate that it spans, is minimal, but that it isn't a basis.

For $k \in K^C$, and $i = 1, ..., 5$, let $z^{k,i}$ denote the sequence defined by, for $n \in \mathbb{N}$ and $j = 1, ..., 5$, $z^{k,i}_{n,j} = \begin{cases} 1 & \text{if n = k \& j = i} \\ 0 & \text{otherwise} \end{cases}$. We now verify that $Z = \{z^{k,i}\}_{k \in K^C, i=1,...5}$ is a minimal spanning set for $X(\mathbf{0}, K)$ but not a basis. Clearly, no element of this set belongs to $X(\mathbf{0}, K)$, since for $n \in K^C$, $z^{k,i}_{n,2} \neq z^{k,i}_{n,3}$. It spans the set however since for an arbitrarily chosen $x \in X(\mathbf{0}, K)$, it is clearly the case that $x = \sum_{j=1,2,4,5} \sum_{n \in K^C} x_{n,j} y^{n,j}$. To show that Z is a minimal spanning set, pick arbitrarily $k \in K^C$, and $i = 1, \ldots, 5$ and omit the sequence $z = z^{k,i}$ from Z. Now consider the "corresponding" member of the basis set $y \in Y$, defined above, where $y = \begin{cases} y^{k,2} & \text{if } i = 3 \\ y^{k,i} & \text{otherise} \end{cases}$. By construction, $y_{k,i} = 1$, but for all $z' \in Z$ except for z, $z'_{k,i} = 0$. Therefore, y cannot be written as a linear combination of the members of Z if z is excluded.

Problem 3 (25 points).

a) Let U denote the set of *all* 2×2 matrices for which no real eigenvalues exist. Is U open, closed or neither? Whatever your answer, specify the appropriate universe. Prove your answer.

Now for the remainder of this question, fix $\beta \in \mathbb{R}_+$ and $\alpha \in \mathbb{R}$ and let $A(\alpha|\beta) = \begin{bmatrix} 2 & \beta \\ \alpha & 4 \end{bmatrix}$. (I'd strongly recommend—who would have thought—that you use a computer to check your answers. If you want to use matlab, the following might help: A = [a , b ; c , d] will define a 2×2 matrix. [Vec,Val] = eig(A) will deliver its eigenvectors and eigenvalues. help eig will give you more details.)

U is a open set in $\mathbb{R}^4$ (or, if you like, $\mathbb{R}^2 \times \mathbb{R}^2$, it doesn't matter). To prove this, consider a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. This matrix will belong to U if there is no value of λ such that the matrix $\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$. has a zero determinant. This will be the case iff $(a+d)^2 - 4(ab-cd) < 0$. But in this case, we can find $\varepsilon > 0$, such that if $A' = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \in B_{\text{Pythag}}(A, \varepsilon)$, then $(a'+d')^2 - 4(a'b'-c'd') < 0$. This establishes that A is an interior element of U.

b) Write down an expression (in terms of α and β) for the eigenvalues of $A(\alpha|\beta)$.

To solve for the eigenvalues, compute the λ's for which the matrix $\begin{bmatrix} 2-\lambda & \beta \\ \alpha & 4-\lambda \end{bmatrix}$. has determinant zero, i.e.,

$$\begin{aligned} (2-\lambda)(4-\lambda) - \alpha * \beta &= 0 \\ \lambda &= 3 \pm \sqrt{1+\alpha\beta} \end{aligned}$$

c) Compute the largest interval I in $\mathbb{R}$ such that $A(\cdot|\beta)$ has real eigenvalues on this interval.

$A(\alpha|\beta)$ will have real eigenvalues provided that $(1+\alpha\beta) \geq 0$, i.e., provided that $\alpha \geq -1/\beta$. Hence $I = [-1/\beta, \infty)$.

d) For $\alpha \in I$, write down an expression for two distinct unit eigenvectors of $A(\alpha|\beta)$.

$\mathbf{v}$ is a unit eigenvector corresponding to λ iff $\mathbf{v} = (v, \sqrt{1-v^2})$ and

$$\begin{bmatrix} 2-\lambda & \beta \\ \alpha & 4-\lambda \end{bmatrix} \begin{bmatrix} v \\ \sqrt{1-v^2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Because the matrix is not invertible, we need to solve for v by substitution. We have

$$\begin{aligned} (\lambda-2)v &= \beta\sqrt{1-v^2} && (5)\\ \alpha v &= (\lambda-4)\sqrt{1-v^2} && (6) \end{aligned}$$

From (5) we have

$$\begin{aligned} v^2 &= 1+\left(\frac{(\lambda-2)v}{\beta}\right)^2 \\ v &= \left[1+\left(\frac{\lambda-2}{\beta}\right)^2\right]^{-0.5} = \frac{\beta}{\sqrt{\beta^2+(2-\lambda)^2}} \end{aligned}$$

Summarizing, the eigenvector corresponding to eigenvalue λ is

$$\mathbf{v} = \left[\tfrac{\beta}{\sqrt{\beta^2+(\lambda-2)^2}}, \tfrac{\lambda-2}{\sqrt{\beta^2+(2-\lambda)^2}}\right]$$

which, substituting for λ

$$\begin{aligned} &= \left[\beta^2+\left(1\underline{+}\sqrt{1+\alpha\beta}\right)^2\right]^{-1/2}\left[\begin{matrix}\beta\\ 1\underline{+}\sqrt{1+\alpha\beta}\end{matrix}\right] \\ &= \left[2+\beta^2+\alpha\beta\underline{+}2\sqrt{1+\alpha\beta}\right]^{-1/2}\left[\begin{matrix}\beta\\ 1\underline{+}\sqrt{1+\alpha\beta}\end{matrix}\right] && (7) \end{aligned}$$

e) Let $\mathbf{v}^1(\alpha)$ and $\mathbf{v}^2(\alpha)$ denote the expressions for the unit eigenvectors you have just calculated and let $\text{Cos}(\alpha)$ denote the cosine of the angle between them. Write down an expression for $\text{Cos}(\alpha)$. (You will be surprised at how simple the expression is.)

From the cosine rule we have

$$\text{Cos}(\alpha) = \frac{\mathbf{v}^1(\alpha)\cdot\mathbf{v}^2(\alpha)}{||\mathbf{v}^1(\alpha)||\;||\mathbf{v}^2(\alpha)||}$$

which, from (7)

$$= \frac{\beta^2+1-(1+\alpha\beta)}{\sqrt{\beta^2+(1+\sqrt{1+\alpha\beta})^2}\cdot\sqrt{\beta^2+(1-\sqrt{1+\alpha\beta})^2}}$$

which, after a great deal of fuss and bother

$$= \frac{(\beta-\alpha)}{\sqrt{4+(\alpha+\beta)^2}} \qquad (8)$$

f) Set $\beta=2$ and (using a computer if you like) sketch a plot of $\text{Cos}(\cdot)$ as a function of α. Interpret your graph in terms of the relationship between the two eigenvectors.

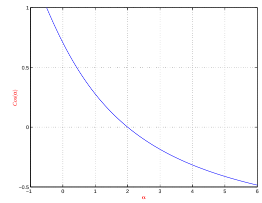

FIGURE 1. Graph of cosine of angle between eigenvectors

Fig. 1 below was generated from the following code.

```
fSymb = '(Beta - Alpha)./sqrt(4 + (Alpha+Beta).2̂)';
Beta = 2;
step = 0.1;
top = 6;
Alpha = -0.5:step:top;
Cos = eval(fSymb);
plot(Alpha,Cos);
grid on  ylabel('Cos(alpha)');
xlabel('alpha');
```

$\text{Cos}(\alpha)$ represents the cosine of the angle between the two eigenvectors. Note that it equals unity, just before the real eigenvectors disappear, i.e., at $\alpha = -1/\beta$, equals zero when $\alpha = \beta$ and is negative thereafter. That is, the two eigenvectors are colinear on the boundary of the interval I, make an acute agnle with each other until $\alpha = \beta$, and an obtuse angle thereafter. Though it's not clear from the graph the angle between the eigenvectors converges to 180 degrees as α increases without bound.

g) Based only on the data you have computed for this question,

i) conjecture a necessary and sufficient condition for a 2×2 matrix to have pairwise orthogonal eigenvectors. For what class of matrix can you *prove* this conjecture, based *only* on results obtained by answering this question?

The conjecture is that the matrix has to be symmetric. You have enough information based on results obtained above to prove that this result holds for all 2×2 matrices with diagonal elements 2 and 4. From (8) you know that regardless of the values of α and β, a necessary and sufficient condition for $\text{Cos}(\alpha) = 0$ is that $\alpha = \beta$.

ii) conjecture one property for the eigenvalues, and one property for the eigenvectors, of a 2×2 matrix which belongs to the boundary of the set of all 2×2 matrices that have real eigenvalues. For what class of matrices do you havea enough information to prove this answer?

A matrix on the boundary has only 2 unit eigenvectors instead of the usual four, and one eigenvalue instead of two.

iii) As a consequence of the answer to part g)ii), the relationship between eigenvectors and non-eigenvectors is fundamentally different for matrices on the above boundary vs matrices in the interior of the set of matrices with real eigenvectors. Explain.

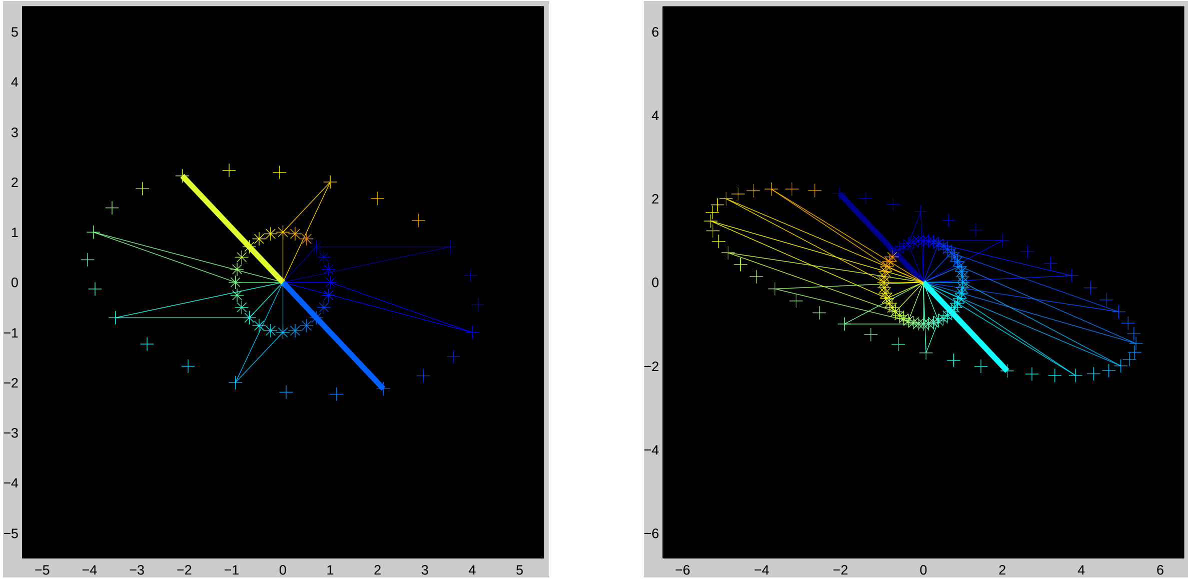

FIGURE 2. Image of unit circle under $A(-1|1)$ vs under B

Because a matrix on the boundary doesn't have "enough" distinct eigenvectors to span '$\mathbb{R}^2$, you cannot write non-eigenvectors as linear combinations of eigenvectors. This has many conseqences: in particular, you cannot determine the shape of the image of the unit circle just by knowing what there is to know about eigenvectors and eigenvalues.

h) Construct 2 distinct 2×2 matrices with the property that (a) their eigenvalues and eigenvectors are identical; (b) the images of the unit circle for the two matrices are different? Hint: look at your answer to part g).

From the answer to the previous question, we might infer that it's possible to construct another matrix on the boundary of U with the above property, and this is in fact the case. Replace the diagonal elements 4 and 2 in the matrix with δ and γ, i.e., consider the abstract matrix. $\begin{bmatrix} \delta & \beta \\ \alpha & \gamma \end{bmatrix}$. We want this matrix to have a unique eigenvalue of 3, and an eigenvector that's $(1/\sqrt{2}, 1/\sqrt{2})$, but we don't want $\delta = 4$. Consider the equation

$$\begin{aligned} 0 &= (\delta-\lambda)(\gamma-\lambda)-\alpha\beta \\ &= \lambda^2-(\delta+\gamma)\lambda+\delta\gamma-\alpha\beta \end{aligned}$$

In order to get properties (a) and (b) above, we need:

$$\begin{aligned} (\delta+\gamma)^2-4(\delta\gamma-\alpha\beta) &= 0 && \text{ensuring that the term under the square root is zero} \\ \delta+\gamma = \lambda &= 3; && \text{giving us a single eigenvalue of 3} \\ \beta &= 3-\delta && \text{giving us an eigenvector } (1/\sqrt{2}, 1/\sqrt{2}) \end{aligned}$$

So we have 4 equations in five unknowns and one degree of freedom: We can find a one-dimensional infinity of matrices with the same eigenvalue and vector as the one above. For example, Consider $B = \begin{bmatrix} 5 & 2 \\ -2 & 1 \end{bmatrix}$. Comparing the image of the unit circle under $A(-1|1)$ to it's image under B we obtain:

Problem 4 (25 points).

Let A be a symmetric $n \times n$ matrix with nonzero eigenvalues $(\lambda_1, \ldots, \lambda_n)$ and consider the difference equation system $\mathbf{x}^t = A\mathbf{x}^{t-1}$, for $t \in \mathbb{N}$. A *solution sequence* for this system is any sequence $(\mathbf{x}^t)_{t=1}^{\infty}$ such that for each t, $\mathbf{x}^t = A\mathbf{x}^{t-1}$. A *steady state* for this system is a solution sequence with the property that all elements of the sequence are equal.

a) Prove the following theorem: a necessary and sufficient condition for the zero sequence to be the *unique* steady state for the system defined by the matrix A is that none of A's eigenvalues is equal to unity.

This is a completely trivial fact! Clearly, the sequence $(\mathbf{x}^t)$, where $\mathbf{x}^t = \mathbf{0}$ for all t, is a steady state since $A\mathbf{0} = \mathbf{0}$. To see that this sequence is the unique steady state iff none of its eigenvalues is equal to zero, pick $\mathbf{x}^t \neq 0$ arbitrarily. If $(\mathbf{x}^t)_{t=1}^{\infty}$ is a steady state, then $\mathbf{x}^2 = \mathbf{x}^1 = A\mathbf{x}^1$. But in this case, $\mathbf{x}^1$ is an eigenvector with a unit eigenvalue.

b) Show that if $|\lambda_i| < 1$, for $i = 1, \ldots, n$, then for any solution sequence, $(\mathbf{x}^t)_{t=1}^{\infty}$, to the system, $\lim_{t \to \infty} \mathbf{x}^t = 0$.

Let $(\mathbf{x}^t)_{t=1}^{\infty}$ be a solution sequence. We'll show that for given $\varepsilon > 0$, there exists T such that $t > T$ implies $||\mathbf{x}^t|| < \varepsilon ||\mathbf{x}^1||$. Since A has full rank, $\mathbf{x}^1$ can be written as a nonnegative linear combination of a selection of its eigenvectors. Choose such a selection, $\mathbf{v}^i$, $i = 1, \ldots n$, such that $\mathbf{x}^1$ belongs to the non-negative cone defined by these eigenvectors, i.e., there exists $\boldsymbol{\alpha} \in \mathbb{R}^n_+$ such that $\mathbf{x}^1 = \sum_{i=1}^n \alpha_i \mathbf{v}^i$ and $\mathbf{x}^2 = \sum_{i=1}^n \alpha_i \lambda_i \mathbf{v}^i$. Now assume that for $t \geq 2$, $\mathbf{x}^t = \sum_{i=1}^n \alpha_i (\lambda_i)^{t-1} \mathbf{v}^i$. (We've just noted that this is true for $t = 2$.) Then $\mathbf{x}^{t+1} = A\mathbf{x}^t = \sum_{i=1}^n \alpha_i A (\lambda_i)^{t-1} \mathbf{v}^i = \sum_{i=1}^n \alpha_i (\lambda_i)^t \mathbf{v}^i$. Since $\boldsymbol{\alpha}$ is nonnegative, it follows that when t is even, $\mathbf{x}^{t+1}$ once again belongs to the nonnegative cone defined by $\mathbf{v}^i$, $i = 1, \ldots n$. In this case, since and $\lambda = \max\{|\lambda_i| : i = 1, \ldots, n\}$, then $||bx^{t+1}|| \leq \lambda^t \sum_{i=1}^n \alpha_i \mathbf{v}^i = \lambda^t ||\mathbf{x}^1||$. Since by assumption $|\lambda| < 1$ we can pick pick t even large enough that $\lambda^t < \varepsilon$, so that $||bx^{t+1}|| \leq \varepsilon ||\mathbf{x}^1||$. We have established therefore that the sequence $(\mathbf{x}^t)_{t=1}^{\infty}$ converges to zero.

c) Now suppose that $|\lambda_1| > |\lambda_i|$, $i = 2, \ldots, n$, and $|\lambda_1| > 1$. Let $\mathbf{v}^1$ be an eigenvector with eigenvalue λ_1. Let $(\mathbf{x}^t)_{t=1}^{\infty}$ be a solution sequence for the system. Characterize the limit behavior of this sequence in terms of the angle θ^t between $\mathbf{x}^t$ and $\mathbf{v}^1$. Your answer should be of the form: $\forall \varepsilon > 0$, there exists T such that if t and $\mathbf{x}^1$ satisfy certain conditions, then some property can be established about this angle. Prove your answer. (Hint: you need to identify a number of different cases. The number is bigger than 2, but smaller than 357.)

As before, choose a selection of pairwise otrhogonal eigenvectors, $\mathbf{v}^i$, $i = 1, \ldots n$, such that for some $\boldsymbol{\alpha} \in \mathbb{R}^n_+$, $\mathbf{x}^1 = \sum_{i=1}^n \alpha_i \mathbf{v}^i$. From the answer to the previous question, we have established that $\mathbf{x}^{t+1} = \sum_{i=1}^n \alpha_i (\lambda_i)^t \mathbf{v}^i$. There are three different cases to consider.

i) Case 1: $\alpha_1 = 0$. Since the vectors $\mathbf{v}^i$, $i = 1, \ldots n$, are all orthogonal to $\mathbf{v}^1$, it follows that for all t

$$\mathbf{x}^{t+1} \cdot \mathbf{v}^1 = \Big(\sum_{i=1}^n \alpha_i (\lambda_i)^t \mathbf{v}^i\Big) \cdot \mathbf{v}^1 = \sum_{i=1}^n \alpha_i (\lambda_i)^t \mathbf{v}^i \cdot \mathbf{v}^1 = 0$$

so that $\theta^t = 90^\circ$ for all t.

ii) Case 2: $\alpha_1 > 0, \lambda_1 > 0$. In this case, again $\mathbf{x}^{t+1} \cdot \mathbf{v}^1 = \sum_{i=1}^n \alpha_i (\lambda_i)^t \mathbf{v}^i \cdot \mathbf{v}^1$. For each t, let $\gamma_1^t = |\alpha_1(\lambda_1)^t|$, and for each i, let $\beta_i^t = \frac{\alpha_i(\lambda_i)^t}{\gamma_1^t}$. Note that $\beta_1^t = 1$, for all t. We can now rewrite the inner product as $\mathbf{x}^{t+1} \cdot \mathbf{v}^1 = \gamma_1^t \sum_{i=1}^n \beta_i^t \mathbf{v}^i \cdot \mathbf{v}^1$. Since $\lambda > 1$ and $|\lambda_1| > |\lambda_i|$, $i = 2, \ldots, n$ it follows that for all $\varepsilon > 0$, there exists T such that for all $t > T$, $|\beta_i^t| < \varepsilon$, for $i = 2, \ldots, n$. It follows that $\sum_{i=1}^n \beta_i^t \mathbf{v}^i \cdot \mathbf{v}^1$ converges to $\mathbf{v}^1 \cdot \mathbf{v}^1$. Hence (θ^t) converges to $0°$.

iii) Case 3: $\alpha_1 > 0, \lambda_1 < 0$. In this case the argument is exactly the same as in case 2, except that $\beta_1^t = (-1)^t$, i.e., alternates between positive and negative 1. Hence (θ^t) has two convergent subsequences, for even and odd values of t. $(\theta^t)_{t=1,3,5}$ converges to $180°$, while $(\theta^t)_{t=2,4,6}$ converges to $0°$.