Final Exam
Wednesday, Oct 21, 2015

This is the final exam for ARE211. As announced earlier, this is an open-book exam. However, use of computers, calculators, Iphones, Ipads, cell phones, Blackberries and other non-human aids with on-off switches is forbidden. Read all questions carefully before starting the test. Allocate your 90 minutes in this exam wisely. The exam has 100 points, so aim for an *average* of roughly 0.9 minutes per point. However, some questions & parts are distinctly easier than others. Make sure that you first do all the easy parts, before you move onto the hard parts. Always bear in mind that if you leave a part-question completely blank, you cannot conceivably get any marks for that part. On the other hand, to give a 100% complete answer to all of the questions would almost certainly take you longer than the time available. In some problems, it may be efficient to simply write down the correct answer (if you know what it is) then come back and justify the answer later.

Problem 1 (True/False) [36 points]:

Answer whether each of the following is true (T), or false (IF). Each part is worth 4 points. Not much more than a small amount of credit will be given for a one (or two) letter answer.

(T) If the statement is *true*, while a rigorous proof is not essential, your credit will increase with the thoroughness of your answer. You don't have to reprove results that were covered in class, but if you cite a theorem taught in class, try to make clear which theorem it is that you are citing.

(F) If you decide that a statement is false, first check if it can be made true by adding an additional condition. Here's an example of what I mean.

Claim: a function f is strictly concave if its Hessian is globally negative definite.

This claim is false as written, but with the following addition it is true:

Claim: a $\mathbb{C}^2$ function f is strictly concave if its Hessian is globally negative definite.

If by adding a condition, a false statement can be made true, you need state what the additional condition is in order to obtain full credit.

While some statements are "fixable" by adding a condition, others cannot be redeemed by adding a condition. For example

Claim: a function f is quasi-concave if each of its lower contour sets is convex

This claim is irredeemably false, since it cannot be made true by adding additional words or conditions. **Do *not* try to convert a false statement to a true one by changing some word/symbol in the statement.** For example, the above Claim would become true if the word "lower" were changed to "upper," but this is not an acceptable modification.

If a statement is *false*, provide a counter-example. For full credit, you must clearly articulate why your counter-example is in fact a counter example, i.e., that is satisfies all properties of the false statement but not the conclusion. Here's an example of what I mean. (Obviously, I'm not going to ask you a question that's quite this deep and subtle.)

Claim: f is twice continuously differentiable implies $f(\cdot) \geq 0$.

This extremely anal answer would be worth full marks, but, pragmatically, could be considered excessively anal.

False. Counter-example: $f(x) = x^3$; $f''(x) = 6x$, which is clearly a continuous function of x, verifying that f is twice continuously differentiable; yet $f(-1) = -1 < 0$, verifying that $f(\cdot)$ is not nonnegative.

A) Consider the following NPP problem: $\max f(\mathbf{x})$ s.t. $g(\mathbf{x}) = \mathbf{b}$, where $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$. If the constraint qualification is satisfied at $\bar{\mathbf{x}}$, then a necessary condition for f to attain a maximum on the constraint set at $\bar{\mathbf{x}}$ is that $g(\bar{\mathbf{x}}) = \mathbf{b}$ and there exists $\boldsymbol{\lambda} \in \mathbb{R}^m_+$ such that $\bigtriangledown f(\bar{\mathbf{x}})=\boldsymbol{\lambda} Jg(\bar{\mathbf{x}})$.

B) Let F denote the set of polynomial functions f mapping $[0,4]$ to $\mathbb{R}$ such that $f(\pi) = c$. Then F is a vector space iff $c = 0$.

C) Let G denote the set of all continuous functions mapping $[0,1]$ to $\mathbb{R}$. G is a vector space.

D) Let G denote the set of all discontinuous functions mapping $[0,1]$ to $\mathbb{R}$. G is a vector space.

E) Given a thrice continuously differentiable function $\xi : \mathbb{R}^{q+p} \to \mathbb{R}^q$, & $(\bar{\mathbf{y}}, \bar{\mathbf{x}}) \in \mathbb{R}^q \times \mathbb{R}^p$, if $\mathrm{Jf}_{\mathbf{y}}(\bar{\mathbf{y}}, \bar{\mathbf{x}})$ is invertible, then there exists a neighborhood X of $\mathbf{x}$ and a unique function ϕ such that for all $\mathbf{x} \in X$, $\xi(\phi(\mathbf{x}), \mathbf{x}) = \xi(\bar{\mathbf{y}}, \bar{\mathbf{x}})$.

F) If $f : X \to Y$ is monotone, then f is both quasi-concave and quasi-convex.

G) If $f : \mathbb{R}^n \to \mathbb{R}$ is both quasi-convex and and quasi-concave, then f is a linear function.

H) Consider the following NPP problem: $\max f(\mathbf{x})$ s.t. $g(\mathbf{x}) \leq \mathbf{b}$, where $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$. Suppose that for all $\mathbf{x}, \mathbf{dx}$, $g(\mathbf{x} + \mathbf{dx}) = g(\mathbf{x}) + Jg(\mathbf{x})\mathbf{dx}$. A necessary condition for f to attain a maximum on the constraint set at $\bar{\mathbf{x}}$ is that $g(\bar{\mathbf{x}}) \leq \mathbf{b}$ and there exists $\boldsymbol{\lambda} \in \mathbb{R}^m_+$ such that $\bigtriangledown f(\bar{\mathbf{x}})=\boldsymbol{\lambda} Jg(\bar{\mathbf{x}})$.

I) If $f : \mathbb{R}^n \to T$ is thrice continuously differentiable, then a sufficient condition for f to be strictly concave is that for all $\mathbf{x}$, $Hf(\mathbf{x})$ is negative definite.

Problem 2 (NPP) [28 points]:
For Coon the cat, eating and sleeping takes up at least 23 hours of the day, sometimes more. During his waking hour, Coon likes to hunt mice and rats. Mice (m) and rats (r) are equally time-consuming to catch: it takes Coon 6 minutes to catch either. But rodent-chasing burns calories, specifically, 10 calories per mouse caught, and, because they are bigger, 20 calories per rat caught, Coon can burn at most 150 calories per day, then he has to go to sleep. (Coon is a vegetarian; he doesn't actually eat the rodents he catches. Moreover, in this world, we allow cats to hunt fractional (but nonnegative) rodents, i.e., Coon's constraint set is a convex set.)

A) [5 points] Draw Coon's constraint set, *with rats on the horizontal axis (for ease of grading).* Label your axes, constraint lines (time and calorie), and label the horizontal and vertical intercepts of the constraint set. Draw the gradient vectors of each constraint. (You're going to add more lines to this graph later, so make sure it's big enough.)

B) [5 points] Coon's utility function is $u_C(m, r) = m + \alpha r$. Write down Coon's programming problem, and the KKT necessary conditions for a solution to his problem.

C) [8 points] Identify conditions on α such that
   (a) His time and calorie constraints are both satisfied with equality, but one of them is not binding. Do each case.
   (b) Coon's eat/sleep constraint is *binding* (not simply satisfied with equality) and his calorie constraint is slack (satisfied with strict inequality)
   (c) Coon's calorie constraint is *binding* (not simply satisfied with equality) and his sleep constraint is slack (satisfied with strict inequality)
   (d) Both constraints are *binding* (not simply satisfied with equality)

D) [5 points] For *one* of the two cases in subpart (b) and for subpart (d) of C), illustrate these cases in on your graph, by drawing an appropriate level set of u_C thru the solution to his problem & the corresponding gradient of u_C. Show graphically that in each case the non-negative cone property of the KKT conditions is satisfied.

E) [5 points] Coon would like to know how his hunting haul would change if he could just get some more nutritious food and thus relax his calorie constraint. His intention is to use the implicit function theorem to solve this problem, but one of his friends tells him that he can't use this theorem for this problem. His friend is of course correct. Explain his friend's reasoning.

Problem 3 (Comparative Statics) [36 points]:
Roberta faces the following constrained maximization problem:

$$\max_x \; x\alpha\beta \quad \text{subject to } x^2+\alpha^2 \le \beta, \quad \text{where } 0<\alpha<\sqrt{\beta}$$

Answers to the computational parts of this question should be in terms of α and β. Don't spend a lot of time trying to get the simplest possible algebraic expression, e.g., it probably wouldn't be worth your time to realize that $\frac{\omega}{\sqrt{\omega^2\gamma}}$ could be more simply written as $\frac{1}{\sqrt{\gamma}}$.

A) [3 points] Write down the KKT necessary condition for Roberta's problem,

B) [3 points] Write down the solution (x^*, λ^*).

C) [8 points] Let $M(\alpha,\beta)$ denote the maximized value of the objective function. Compute, to a first order approximation, how M changes when α and β change?

D) [2 points] Interpret λ^*. (**Hint:** In the immortal words of Phaedrus, "Things are not always as they seem; the first appearance deceives many.")

E) [3 points] **Bonus**. Interpret/explain/elucidate the hint in the previous part.

F) [14 points] *Without taking derivatives of the expression for x^* that you obtained in B)*, compute $\frac{dx^*}{d\alpha}$ and $\frac{dx^*}{d\beta}$. *Show all your work!*

G) [3 points] You will have noticed that you could have solved part F) *much* more quickly had you been allowed to differentiate your answer to part B). This raises the question: why was the theorem that you used ever invented in the first place? Comment.

H) [3 points] Roberta is given a one-time opportunity to purchase, for one dollar, either an additional unit of α or an additional unit of β. Identify a condition relating α to β that will determine whether she will choose to purchase α rather than β.