Fall2015 **ARE211**

Final Exam - Answer key

Problem 1 (True/False) [36 points]:

Answer whether each of the following is true (T), or false (IF). Each part is worth 4 points. Not much more than a small amount of credit will be given for a one (or two) letter answer.

(T) If the statement is *true*, while a rigorous proof is not essential, your credit will increase with the thoroughness of your answer. You don't have to reprove results that were covered in class, but if you cite a theorem taught in class, try to make clear which theorem it is that you are citing.

(F) If you decide that a statement is false, first check if it can be made true by adding an additional condition. Here's an example of what I mean.

Claim: a function f is strictly concave if its Hessian is globally negative definite.

This claim is false as written, but with the following addition it is true:

Claim: a $\mathbb{C}^2$ function f is strictly concave if its Hessian is globally negative definite.

If by adding a condition, a false statement can be made true, you need state what the additional condition is in order to obtain full credit.

While some statements are "fixable" by adding a condition, others cannot be redeemed by adding a condition. For example

Claim: a function f is quasi-concave if each of its lower contour sets is convex

This claim is irredeemably false, since it cannot be made true by adding additional words or conditions. **Do *not* try to convert a false statement to a true one by changing some word/symbol in the statement.** For example, the above Claim would become true if the word "lower" were changed to "upper," but this is not an acceptable modification.

If a statement is *false*, provide a counter-example. For full credit, you must clearly articulate why your counter-example is in fact a counter example, i.e., that is satisfies all properties of the false statement but not the conclusion. Here's an example of what I mean. (Obviously, I'm not going to ask you a question that's quite this deep and subtle.)

Claim: f is twice continuously differentiable implies $f(\cdot) \geq 0$.

This extremely anal answer would be worth full marks, but, pragmatically, could be considered excessively anal.

False. Counter-example: $f(x) = x^3$; $f''(x) = 6x$, which is clearly a continuous function of x, verifying that f is twice continuously differentiable; yet $f(-1) = -1 < 0$, verifying that $f(\cdot)$ is not nonnegative.

A) Consider the following NPP problem: $\max f(\mathbf{x})$ s.t. $g(\mathbf{x}) = \mathbf{b}$, where $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$. If the constraint qualification is satisfied at $\bar{\mathbf{x}}$, then a necessary condition for f to attain a maximum on the constraint set at $\bar{\mathbf{x}}$ is that $g(\bar{\mathbf{x}}) = \mathbf{b}$ and there exists $\boldsymbol{\lambda} \in \mathbb{R}^m_+$ such that $\bigtriangledown f(\bar{\mathbf{x}})=\boldsymbol{\lambda} Jg(\bar{\mathbf{x}})$.

Ans: (F). When the constraint set must be satisfied with equality then the KKT requirement is $\boldsymbol{\lambda} \in \mathbb{R}^m$, i.e., λ can be negative. For example consider a standard utility maximization problem, where the consumer's indifference curves are represented by circles and her utility is maximized at zero. In this case, the solution to the above problem would require a tangency between her indifference curves and the budget line, i.e., the gradient of the constraint and the gradient of the objective would point in opposite directions, hence the lagrangian would be negative.

B) Let F denote the set of polynomial functions f mapping $[0,4]$ to $\mathbb{R}$ such that $f(\pi) = c$. Then F is a vector space iff $c = 0$.

Ans: (T). Let f, g be polynomials satisfying $f(\pi) = g(\pi) = 0$. Then for $\alpha, \beta \in \mathbb{R}$, $h(\pi) = \alpha f(\pi) + \beta g(\pi) = 0$. Therefore, $h \in F$. On the other hand, suppose that $f(\pi) = c \neq 0$. Let $g = f$, and let $\alpha = \beta = 1$. Then $h(\pi) = \alpha f(\pi) + \beta g(\pi) = 2c$, Therefore, $h \notin F$.

C) Let G denote the set of all continuous functions mapping $[0,1]$ to $\mathbb{R}$. G is a vector space.

Ans: (T). Let $f : [0,1] \to \mathbb{R}$ and $g : [0,1] \to \mathbb{R}$, and pick $\alpha, \beta \in \mathbb{R}$. Then $h = \alpha f + \beta g$ maps $[0,1]$ to $\mathbb{R}$ and is continuous. Hence G is a vector space.

D) Let G denote the set of all discontinuous functions mapping $[0,1]$ to $\mathbb{R}$. G is a vector space.

Ans: (F). Let $f(x) = \begin{cases} 0 & x \leq 1 \\ 1 & x > 1 \end{cases}$ and let $g = 1 - f$. Clearly $f, g \in G$. However, let $h = f + g$. Clearly $h(\cdot) = 1$ and hence is continuous, i.e., $h \notin G$. Hence G is not a vector space.

E) Given a thrice continuously differentiable function $\xi : \mathbb{R}^{q+p} \to \mathbb{R}^q$, & $(\bar{\mathbf{y}}, \bar{\mathbf{x}}) \in \mathbb{R}^q \times \mathbb{R}^p$, if $\mathrm{Jf}_{\mathbf{y}}(\bar{\mathbf{y}}, \bar{\mathbf{x}})$ is invertible, then there exists a neighborhood X of $\mathbf{x}$ and a unique function ϕ such that for all $\mathbf{x} \in X$, $\xi(\phi(\mathbf{x}), \mathbf{x}) = \xi(\bar{\mathbf{y}}, \bar{\mathbf{x}})$.

Ans: (F-M). the following is true: Given a thrice continuously differentiable function $\xi : \mathbb{R}^{q+p} \to \mathbb{R}^q$, & $(\bar{\mathbf{y}}, \bar{\mathbf{x}}) \in \mathbb{R}^q \times \mathbb{R}^p$, if $\mathrm{Jf}_{\mathbf{y}}(\bar{\mathbf{y}}, \bar{\mathbf{x}})$ is invertible, then there exists a neighborhood X of $\mathbf{x}$, a neighborhood Y of $\mathbf{y}$, and a unique function $\phi :\ X \to Y$ such that for all $\mathbf{x} \in X$, $\xi(\phi(\mathbf{x}), \mathbf{x}) = \xi(\bar{\mathbf{y}}, \bar{\mathbf{x}})$.

F) If $f : X \to Y$ is monotone, then f is both quasi-concave and quasi-convex.

Ans: (F-M). As written the statement is not true: for a counter-example, let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = x^2+y^2$. f is convex but is not quasi-concave, for example: $\{x, y : f(x,y) \geq 1\}$ is not a convex set. The statement would be true if we added $X \subset \mathbb{R}$. In this case, fix $\alpha \in Y$ and $x \in X$ such that $f(x) = \alpha$. Then the upper-contour set corresponding to α is $[x, \infty)$ which is convex. Similarly, the lower-contour set corresponding to α is $(\infty, x]$ which is convex. Hence f is both quasi-concave and quasi-convex.

G) If $f : \mathbb{R}^n \to \mathbb{R}$ is both quasi-convex and and quasi-concave, then f is a linear function.

Ans: (F). in addition to linear functions, functions that are affine but not linear are also both quasi-convex and and quasi-concave.

H) Consider the following NPP problem: $\max f(\mathbf{x})$ s.t. $g(\mathbf{x}) \leq \mathbf{b}$, where $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$. Suppose that for all $\mathbf{x}, \mathbf{dx}$, $g(\mathbf{x} + \mathbf{dx}) = g(\mathbf{x}) + Jg(\mathbf{x})\mathbf{dx}$. A necessary condition for f to attain a maximum on the constraint set at $\bar{\mathbf{x}}$ is that $g(\bar{\mathbf{x}}) \leq \mathbf{b}$ and there exists $\boldsymbol{\lambda} \in \mathbb{R}^m_+$ such that $\bigtriangledown f(\bar{\mathbf{x}}) = \boldsymbol{\lambda} Jg(\bar{\mathbf{x}})$.

Ans: (T). For the general NPP problem, the conditions given are necessary only if the constraint qualification is known to be satisfied. But in this case, for every $\mathbf{x}, \mathbf{dx}$, $g(\mathbf{x} + \mathbf{dx}) = g(\mathbf{x}) + Jg(\mathbf{x})\mathbf{dx}$, which means that all higher order terms in all Taylor expansions are zero. Hence the function g is affine. The constraint qualification is that the linearized version of the constraint set is locally equivalent to the original version. When g is affine, the linearized version of the constraint set and the original version coincide. Hence the CQ is vacuously satisfied.

I) If $f : \mathbb{R}^n \to T$ is thrice continuously differentiable, then a sufficient condition for f to be strictly concave is that for all $\mathbf{x}$, $Hf(\mathbf{x})$ is negative definite.

Ans: (F-M). for $m > 1$, the statement is nonsense, since concavity is defined only for scalar-valued functions. If we add the caveat $m = 1$ to the statement, it becomes true.

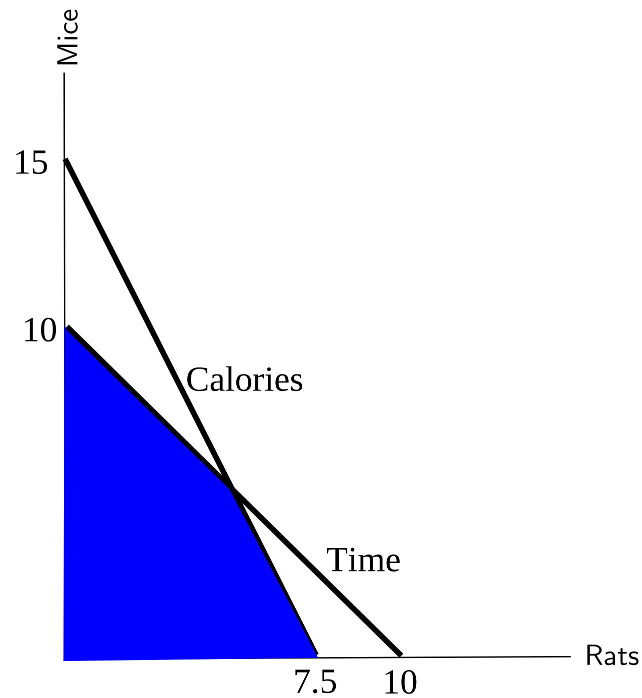

FIGURE 1. Coon's Constraint Set

Problem 2 (NPP) [28 points]:
For Coon the cat, eating and sleeping takes up at least 23 hours of the day, sometimes more. During his waking hour, Coon likes to hunt mice and rats. Mice (m) and rats (r) are equally time-consuming to catch: it takes Coon 6 minutes to catch either. But rodent-chasing burns calories, specifically, 10 calories per mouse caught, and, because they are bigger, 20 calories per rat caught, Coon can burn at most 150 calories per day, then he has to go to sleep. (Coon is a vegetarian; he doesn't actually eat the rodents he catches. Moreover, in this world, we allow cats to hunt fractional (but nonnegative) rodents, i.e., Coon's constraint set is a convex set.)

A) [5 points] Draw Coon's constraint set, *with rats on the horizontal axis (for ease of grading)*. Label your axes, constraint lines (time and calorie), and label the horizontal and vertical intercepts of the constraint set. Draw the gradient vectors of each constraint. (You're going to add more lines to this graph later, so make sure it's big enough.)

Ans: See Fig. 1.

B) [5 points] Coon's utility function is $u_C(m,r) = m + \alpha r$. Write down Coon's programming problem, and the KKT necessary conditions for a solution to his problem.

Ans: In this question I wrote down the constraints in rodent units, i.e., because the of time constraint Coon can't catch more than 10 rodents; because of the calorie constraint he can't catch more than 15 mice or 7.5 rats. Many of you wrote down the answer in minutes units. Both approaches are equally acceptable, since the choice of rodent vs minute units affects the *length* of the gradients of the constraints, but not their *directions*. Since it's only their directions that define non-negative cones, directions are all we need for the purposes of determining which coefficients are binding/slack/satisfied with equality but not binding. The magnitude of the λ's will also depend on units, as in how much more utility does Coon get from an extra minute vs an extra rodent.

Coon's programming problem is

$$\max_{m,r} \; m + \alpha r \quad \text{subject to } m + r \leq 10; m + 2r \leq 15; m \geq 0; r \geq 0$$

Because Coon's payoff function is strictly monotone, the nonnegativity constraints are always slack, so I'm going to ignore them. His KKT necessary conditions are

$$\begin{aligned} \begin{bmatrix} 1 & \alpha \end{bmatrix} &= \begin{bmatrix} \lambda_{\text{time}} & \lambda_{\text{cal}} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \\ m + 2r < 15 &\implies \lambda_{\text{cal}} = 0 \\ m + r < 10 &\implies \lambda_{\text{time}} = 0 \end{aligned}$$

C) [8 points] Identify conditions on α such that

(a) His time and calorie constraints are both satisfied with equality, but one of them is not binding. Do each case.

Ans: His calorie constraint binds and his time constraint is satisfied with equality but not binding iff $\alpha = 2$

His time constraint binds and his calorie constraint is satisfied with equality but not binding iff $\alpha = 1$

(b) Coon's eat/sleep constraint is *binding* (not simply satisfied with equality) and his calorie constraint is slack (satisfied with strict inequality)

Ans: $\alpha \in (0, 1)$. He doesn't particularly care for rats, and they require more calories, so he doesn't bother to catch any rats at all.

(c) Coon's calorie constraint is *binding* (not simply satisfied with equality) and his sleep constraint is slack (satisfied with strict inequality)

Ans: $\alpha > 2$. He likes rats *much* more than mice, so he doesn't bother to catch any mice at all.

(d) Both constraints are *binding* (not simply satisfied with equality)

Ans: $\alpha \in (1, 2)$. He likes rats a little more than mice, but not so much more that he's willing to give up entirely on mice.

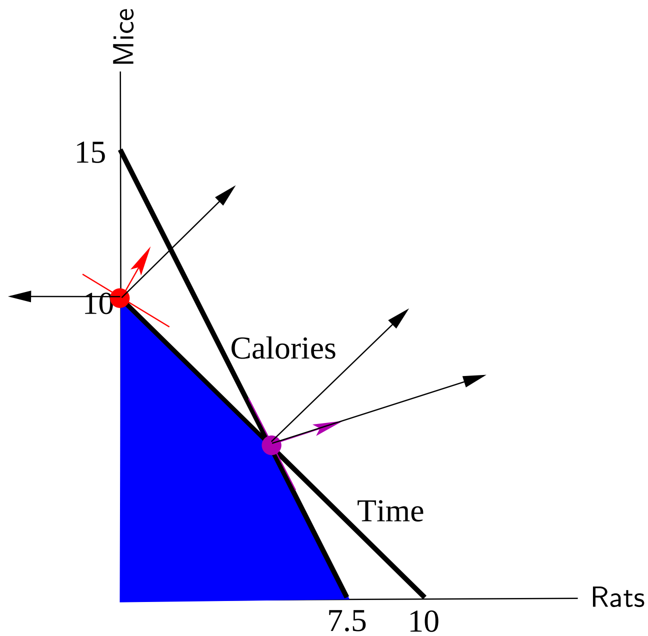

FIGURE 2. Coon's Constraint Set

D) [5 points] For *one* of the two cases in subpart (b) and for subpart (d) of C), illustrate these cases in on your graph, by drawing an appropriate level set of u_C thru the solution to his problem & the corresponding gradient of u_C. Show graphically that in each case the non-negative cone property of the KKT conditions is satisfied.

Ans: See Fig. 2.

E) [5 points] Coon would like to know how his hunting haul would change if he could just get some more nutritious food and thus relax his calorie constraint. His intention is to use the implicit function theorem to solve this problem, but one of his friends tells him that he can't use this theorem for this problem. His friend is of course correct. Explain his friend's reasoning.

Ans: Since Coon's objective and constraints are all linear functions, the Hessian of the Lagrangian is zero, so that the non-zero determinant requirement of the implicit function theorem fails.

Problem 3 (Comparative Statics) [36 points]:
Roberta faces the following constrained maximization problem:

$$\max_x \; x\alpha\beta \quad \text{subject to } x^2 + \alpha^2 \leq \beta, \quad \text{where } 0 < \alpha < \sqrt{\beta}$$

Answers to the computational parts of this question should be in terms of α and β. Don't spend a lot of time trying to get the simplest possible algebraic expression, e.g., it probably wouldn't be worth your time to realize that $\frac{\omega}{\sqrt{\omega^2\gamma}}$ could be more simply written as $\frac{1}{\sqrt{\gamma}}$.

A) [3 points] Write down the KKT necessary condition for Roberta's problem,

Ans: $\bigtriangledown f = \alpha\beta$, $\bigtriangledown g = 2x$ so the KKT condition is $\alpha\beta = 2\lambda x$; if $x^2 + \alpha^2 < \beta$, then $\lambda = 0$.

B) [3 points] Write down the solution (x^*, λ^*).

Ans: $x^* = \sqrt{\beta - \alpha^2}$; $\lambda^* = \frac{\alpha\beta}{2x^*} = \frac{\alpha\beta}{2\sqrt{\beta-\alpha^2}}$.

C) [8 points] Let $M(\alpha, \beta)$ denote the maximized value of the objective function. Compute, to a first order approximation, how M changes when α and β change?

Ans:

$$\begin{aligned}
\frac{\partial M(\alpha,\beta)}{\partial \alpha} &= \frac{\beta(\beta - 2\alpha^2)}{\sqrt{\beta - \alpha^2}} \\
\frac{\partial M(\alpha,\beta)}{\partial \beta} &= \frac{\alpha(\alpha^2 - 1.5\beta)}{\sqrt{\beta - \alpha^2}} \\
M(\alpha + d\alpha, \beta + d\beta) &\approx \begin{bmatrix} \frac{\partial M(\alpha,\beta)}{\partial \alpha} & \frac{\partial M(\alpha,\beta)}{\partial \beta} \end{bmatrix} \begin{bmatrix} d\alpha \\ d\beta \end{bmatrix}
\end{aligned}$$

D) [2 points] Interpret λ^*. (**Hint:** In the immortal words of Phaedrus, "Things are not always as they seem; the first appearance deceives many.")

Ans: λ^* is the ratio of the partial derivative w.r.t. x of the objective function to the partial derivative w.r.t. x of the constraint function, where both derivatives are evaluated at the solution to the problem. It is *not* the shadow value of the constraint. See answer to the next part.

E) [3 points] **Bonus**. Interpret/explain/elucidate the hint in the previous part.

Ans: In the standard presentation of the KKT conditions, λ^* would be the shadow value of the constraint. But in the standard presentation, the objective does *not* depend on the value of the level set of the constraint, in this case β. Since in this problem, β is also an argument of the objective function, the standard interpretation does not apply.

F) [14 points] *Without taking derivatives of the expression for x^* that you obtained in B),* compute $\frac{dx^*}{d\alpha}$ and $\frac{dx^*}{d\beta}$. *Show all your work!*

Ans: The level set to which we apply the the implicit function theorem is

$$\begin{aligned} \alpha\beta - 2\lambda x &= 0 \\ \beta - x^2 - \alpha^2 &= 0 \end{aligned}$$

thus

$$\begin{bmatrix} \frac{\partial x}{\partial \alpha} & \frac{\partial x}{\partial \beta} \\ \frac{\partial \lambda}{\partial \alpha} & \frac{\partial \lambda}{\partial \beta} \end{bmatrix} = - \begin{bmatrix} -2\lambda & -2x \\ -2x & 0 \end{bmatrix}^{-1} \begin{bmatrix} \beta & \alpha \\ -2\alpha & 1 \end{bmatrix}$$

applying Cramer's Rule

$$\begin{aligned} \frac{\partial x}{\partial \alpha} &= - \det\left(\begin{bmatrix} \beta & -2x \\ -2\alpha & 0 \end{bmatrix}\right) \Big/ \det\left(\begin{bmatrix} -2\lambda & -2x \\ -2x & 0 \end{bmatrix}\right) \\ &= -\frac{\alpha}{x} = -\frac{\alpha}{\sqrt{\beta - \alpha^2}} \\ \frac{\partial x}{\partial \beta} &= - \det\left(\begin{bmatrix} \alpha & -2x \\ 1 & 0 \end{bmatrix}\right) \Big/ \det\left(\begin{bmatrix} -2\lambda & -2x \\ -2x & 0 \end{bmatrix}\right) \\ &= \frac{1}{2x} = \frac{1}{2\sqrt{\beta - \alpha^2}} \end{aligned}$$

G) [3 points] You will have noticed that you could have solved part F) *much* more quickly had you been allowed to differentiate your answer to part B). This raises the question: why was the theorem that you used ever invented in the first place? Comment.

Ans: In this particular example you were able to solve explicitly for the solution. In general this will not be the case, indeed, in general, an explicit expression for the solution won't exist. The implicit function theorem works generally (except when the Jacobian condition fails), so in many (most) cases, it's the only way to obtain the answer.

H) [3 points] Roberta is given a one-time opportunity to purchase, for one dollar, either an additional unit of α or an additional unit of β. Identify a condition relating α to β that will determine whether she will choose to purchase α rather than β.

Ans: (To a first order approximation), she'll purchase α rather than β iff $\frac{\partial M(\alpha,\beta)}{\partial \alpha} > \frac{\partial M(\alpha,\beta)}{\partial \beta}$. From part C), this property holds iff $\beta(\beta - 2\alpha^2) > \alpha(\alpha^2 - 1.5\beta)$