Potential Final Exam Solutions

Real Analysis

1. Let $a \in A$, where A is an open set. Given some sequence a_n converging to a, show that all but a finite number of the terms of a_n must be contained in the set A.

Solution: Let $a \in A$, where A is an open set. Let $a = lima_n$. It follows that there exists an epsilon ball around a such that $b_\epsilon(a) \in A$. Because a_n is a convergent sequence, every epsilon ball around a contains all but a finite number of the terms of a_n. Therefore, all but a finite number of the terms of a_n must be contained in the set A.

2. Let a be an accumulation point of the set $B \cup C$. Show that a is either an accumulation point of B or C, or both sets.

a. Prove that $cl(B \cup C) = cl(B) \cup cl(C)$.

b. Extra Credit: Can you extend (b) to an infinite union of closures? In other words, does $cl(\bigcup_i^\infty B_i) = cl(B_1) \cup cl(B_2) \cup ...$?

Solution: (a) Since a be an accumulation point, there exists some sequence a_n that converges to a, where $a_n \neq a$ for all n. Since a_n is contained in $B \cup C$, at least one must contain an infinite number of the terms of a_n, though both could. A subsequence of these terms must also be converge to a.

(b) Since $cl(A) \subset cl(A \cup B)$, $cl(B) \subset cl(A \cup B)$, we have $cl(A) \cup cl(B) \subset cl(A \cup B)$. We also know that $A \cup B \subset cl(A) \cup cl(B)$, so $cl(A \cup B) \subset cl(cl(A) \cup cl(B))$. $cl(A) \cup cl(B)$ is closed, so $cl(A \cup B) = cl(cl(A) \cup cl(B))$. So $cl(A \cup B) \subset cl(A) \cup cl(B)$, so $cl(A) \cup cl(B) = cl(A \cup B)$.

(c) No. Let $B_n = 1/n$. Then $cl(\bigcup_i^\infty B_i) = \{1/n\} \cup \{0\}$ but $cl(B_1) \cup cl(B_2) \cup ... = \{1/n\}$.

3. Prove a finite set is (i) closed, (ii) bounded, and (iii) admit a finite subcover.

Solution: (i) Let $X = \{x_1, ..., x_n\}$ be a finite set. Pick any point c in the complement of X. Let the set D be the set of distances from the points in X and c. Let $\epsilon = minD$. The epsilon ball around c (of size ϵ) is contained in the compliment of X. So the compliment of X is open, therefore X is closed.

(ii) Suppose X is an a universe endowed with the metric d. Let Z be the set of distances from X to zero. The set is bounded by $max(Z)$.

(iii) Let $X = \{x_1, ..., x_n\}$ be a finite set with an open cover $\mathbb{O}$. This means for each $x_i \in X$ there is an open set $O_i \in \mathbb{O}$ such that $x_i \in O_i$. Therefore X is contained in $O_1 \cup ... \cup O_n$.

Linear Algebra

1. Let $S = \{(x_1, x_2) | x_1, x_2 \in \mathbb{R}\}$. If (x_1, x_2), $(y_1, y_2) \in S$ and $c \in \mathbb{R}$, define $(x_1, x_2) + (y_1, y_2) = (x_1 + 2y_1, x_2 + 3y_2)$ and $c(x_1, x_2) = (cx_1, cx_2)$. Is S a vector space with these operations? Why or why not.

Solution: Not a vector space. Since $x + y \neq y + x$ if $y \neq x$, commutativity of addition does not hold.

2. Let $M = \begin{bmatrix} m & 1 \\ 1 & m \end{bmatrix}$. Under what conditions on m is the matrix M invertible?

(a) For each value of m that makes M non-invertible, determine the rank of M.

Solution: By row reduction (row 2 minus m times row 1):

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 - m^2 \end{bmatrix}.$$

Which is invertible if and only if $1 - m^2 \neq 0$. So $m \neq 1$, $m \neq -1$. For each of those m values, the matrix has rank 1.

3. True or False. If true, show why. If false, provide a counterexample.

(a) If x and y are linearly independent, and if $\{x; y; z\}$ is linearly dependent, then z is in the span of $\{x; y\}$.

Solution: True. Since x and y are linearly independent, and x, y, z is linearly dependent, by definition, z can be expressed as a linear combination of x and y, and therefore z is in the span of x and y.

(b) Let $A = \{a_1, ..., a_j\}$ where $a_i \in \mathbb{R}^k$ for $i = 1, ...j$. If $j < k$ then A is a linearly independent set.

Solution: False. $A = \{[1, 0, 0], [2, 0, 0]\}$ is a set of linear dependent vectors.

(c) If A is a linearly dependent set of vectors, then each vector in A is a linear combination of the other vectors in A.

Solution: Consider $A = \{[1, 0, 0], [0, 1, 0], [0, 2, 0]\}$. Though this is a set of linearly dependent vectors, the first vector cannot be expressed as a linear combination of the other vectors.

(d) If A is symmetric, then A is invertible.

False: Let A be a 2x2 matrix of zeros.

(e) If A is a $2x2$ matrix such that $A(Ax) = 0$ for all $x \in \mathbb{R}$, then A is the zero matrix.

Solution: False. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

(f) If V is a vector space and F is a finite set of vectors in V , then some subset of F forms a basis for V.

Solution: False. Let $V = \mathbb{R}^3$ and $F = \{(1, 0, 0), (0, 1, 0)\}$.

(g) The set of polynomials of degree $\leq k$, where $k \in \mathbb{N}$ is a vector space. (Reminder: an k'th degree polynomial is a function of the form $f(x) = \sum_{i=0}^{k} a_i x^i$, where $(a_0, ..., a_n) \in \mathbb{R}^{\ltimes+\nparallel}$ and $a_n \neq 0$

Solution: True. It is closed under addition and scaler multiplication.

Calculus

1. Let $f : \mathbb{R}^3 \to \mathbb{R}$, $f = xy^2z^3 + ze^{x^2y}$

(a) what is the gradient of f?

Solution: $\nabla f = \begin{bmatrix} y^2z^3 + 2xze^{x^2y} \\ 2xyz^3 + zx^2e^{x^2y} \\ 3xy^2z^2 + e^{x^2y} \end{bmatrix}$

(b) what is the directional derivative of f, f_h, at $(1, 2, 3)$ when $h = (3, 1, 4)$?

Solution: $\nabla f(1, 2, 3) \cdot \frac{h}{||h||}$.

(c) Now let $y = x^{1/2}$ and $z = x^{1/6}$. What is the total derivative of f with respect to x. Use this to approximate the change in f when x increases by 0.5 from $(1, 2, 3)$.

Solution: $\frac{\delta f}{\delta x} = f_x + f_y \frac{\delta y}{\delta x} + f_z \frac{\delta z}{\delta x}$.

(d) Rewrite the function f as a function $h : \mathbb{R} \to \mathbb{R}$. What is $h'(x)$? Use this derivative to approximate the change in f when x increases by 0.5 from $(1, 2, 3)$.

Solution: $h(x) = x(x^{1/2})^2(x^{1/6})^3 + (x^{1/6})e^{x^2(x^{1/2})}$

$h(x) = x^{5/2} + x^{1/6}e^{x^{3/2}}$

2. Suppose $f : \mathbb{R} \to \mathbb{R}$ is k times differentiable and $k > 1$. Suppose that $f^{'}(x_0) = ... = f^{(k-1)}(x_0) = 0$ and $f^{(k)}(x_0) \neq 0$. Prove:

(a) if k is even and $f^{(k)}(x_0) < 0$, then f attains a is a strict local max at x_0.

Solution: Since k is even and $f^{(k)}(x_0) < 0$, then $T_n < 0$ for all dx. By the Taylor Young theorem, the absolute value of the expansion dominates the remainder. So $f(x + dx) < f(x)$ for all dx.

(b) if k is even and $f^{(k)}(x_0) > 0$, then f attains a is a strict local min at x_0.

Solution: Since k is even and $f^{(k)}(x_0) > 0$, then $T_n > 0$ for all dx. By the Taylor Young theorem, the absolute value of the expansion dominates the remainder. So $f(x + dx) > f(x)$ for all dx.

(c) if k is odd, x_0 is not necessarily a local max nor a local min.

Solution: If k is odd, then you do not know the sign of $(x - x_0)^n$.

NPP

1. Minimize $x^2 + y^2 - 4xy - \beta x$ subject to $4x + 3y \leq 10$, $y - 4x^\alpha \geq -2$, $x \geq 0$, $y \geq 0$ $(\alpha, \beta \in \mathbb{R})$.

(a) Does a solution exist for this problem? Explain why or why not.

Solution: A solution to this problem does exist. The objective function is continuous and the constraint set is compact, so the extreme value theorem guarantees a solution.

(b) If a solution exists, how could you be sure that the solution is unique?

Solution: Theorem 2.

(c) Write the problem in the canonical NPP format.

Solution:

Maximize $-x^2 - y^2 + 4xy + \beta x$ subject to $4x + 3y \leq 10$, $-y + 4x^\alpha \leq 2$, $-x \leq 0$, $-y \leq 0$.

(d) If $\alpha = 1$, explain why the constraint qualification is satisfied at all points in language that involves no mathematical symbols or the phrase "linear independence". Show the constraint qualification holds at all potential solutions if $\alpha = 2$?

Solution: The constraint qualification is satisfied at a point if the linearized version of the constraint set is, in a neighborhood of that point, almost identical to the original constraint set. The constraints are linear, so the linearized version of the constraints is identical to the constraint set at every point in the constraint set.

If $\alpha = 2$,

$$CQ = \begin{bmatrix} 43 \\ 8x - 1 \end{bmatrix}$$

(e) Assume $\beta = 9$, $\alpha = 2$. Write the lagrangian, derive all of the first-order conditions to the NPP problem, and solve.

Solution:

$$\mathbb{L} = -x^2 - y^2 + 4xy + 9x + \lambda_1()$$

If $\lambda_1 > 0$ and $\lambda_2 = 0$ then $x = 28/37$, $y = 86/37$, and $\lambda_1 = 14/37$.

(f) Does the gradient vanish at any points on the constraint set?

Solution:

$$\nabla f = \begin{bmatrix} -2x + 4y + 9 \\ -2y + 4x \end{bmatrix}$$

The gradient vanishes if $x = frac12y$ and $-2(frac12y) + 4(frac12y) + 9 = 0$. So $-y + 2y + 9 = 0$, and this only occurs at $y = -9$, which is outside of the constraint set.

(g) How would you check the second-order conditions? (Just describe the process, and if you would use any matrices or equations, write those out).

Solution: Check the principal minors of the bordered hessian.

(h) Apply the Envelope Theorem to estimate the solution to the NPP problem where $\beta = 9.2$.

Solution: $\frac{\delta L}{\delta \beta} = x$. So $M(\beta + d\beta) \approx M(\beta) + x(0.2) = -(28/37)^2 - (86/37)^2 + 4(28/37)y + 9(28/37) + (28/37)(0.2)$

Envelope Theorem/Implicit Function Theorem

1. Consider the equation $x - e^y + z^2 = 4$. Define $f(x, y, z) = x - e^y + z^2 - 4$

(a) Does the equation define z as an implicit function of x, y at $(x, y) = (6, 0)$?

Solution: $6 - e^0 + z^2 = 4$, $5 + z^2 = 4$, $z^2 = -1$, so there is no real solution.

(b) Does the equation define z as an implicit function of x, y at $(x, y) = (5, 0)$? Can you write z as an implicit function in a neighborhood of $(x, y) = (5, 0)$.

Solution: $5 - e^0 + z^2 = 4$, $4 + z^2 = 4$, $z^2 = 0$, $z = 0$. But, at $(x, y, z) = (5, 0, 0)$, $\frac{\delta f}{\delta z} = 2z = 0$, so we cannot use the IFT.

(c) Does the equation define z as an implicit function of x, y at $(x, y) = (-4, 0)$?

Solution: $f(-6, 0, z) = -4 - 1 + z^2 - 4 \rightarrow z = \pm 3$. At $(-6, 0, \pm 3)$, $\frac{\delta f}{\delta z} \neq 0$.

At $z = 3$,

$[\frac{\delta x}{\delta y}] = -\frac{\delta f^{-1}}{\delta z}[\frac{\delta f}{\delta x} \quad \frac{\delta f}{\delta y}]$

$= \frac{-1}{6}[1 \quad -e^y] = \frac{-1}{6}[1 \quad -1] = [\frac{-1}{6} \quad \frac{1}{6}]$

(d) What is the value of z that corresponds to $(x, y) = (-6.2, 0.2)$.

Solution: $g((-6, 0) + (-0.2, 0.2)) \approx g((-6, 0)) + \frac{\delta g}{\delta x}\delta x + \frac{\delta g}{\delta y}\delta x$

$= 3 + (\frac{-1}{6}(-0.2)) + (\frac{1}{6}(0.2))$