**Fall2012** **ARE211**

FINAL EXAM

WEDNESDAY, DEC 12, 2012

This is the final exam for ARE211. As announced earlier, this is an open-book exam. However, use of computers, calculators, Iphones, Ipads, cell phones, Blackberries and other non-human aids with on-off switches is forbidden. Read all questions carefully before starting the test. Allocate your 180 minutes in this exam wisely. The exam has 100 points, so aim for an *average* of roughly 1.8 minutes per point. However, some questions & parts are distinctly easier than others. Make sure that you first do all the easy parts, before you move onto the hard parts. Always bear in mind that if you leave a part-question completely blank, you cannot conceivably get any marks for that part. On the other hand, to give a 100% complete answer to all of the questions would almost certainly take you longer than the time available. In some problems, it may be efficient to simply write down the correct answer (if you know what it is) then come back and justify the answer later.

**Problem 1 (Calculus) [24 points]:**

Compute the partial derivatives of the following functions and then determine whether or not they are differentiable at (0,0). Justify your answers.

A) [12 points] $f(x,y) = (x^2y)^{\frac{1}{3}}$.

B) [12 points] $f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right), & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$

**Problem 2 (Constrained Optimization) [24 points]:**

Consider the following constrained maximization problem.

$$\begin{aligned} &\max x+y \\ \text{subject to } &g1 : x^2 - y \leq 3 \\ &g2 : x^2 + y \leq 5 \end{aligned}$$

A) [4 points] Sketch the constraint set and draw a level set of the utility function.

B) [4 points] Does a solution to the problem exist? Explain.

C) [6 points] Write down the Lagrange and the KKT conditions.

D) [10 points] Find the solution to the problem. Show your work for all cases for maximum credit.

**Problem 3 (Comparative Statics) [20 points]:**
Consider the following constrained maximization problem.

$$\max x + \alpha \text{ subject to } x^2 + \alpha \leq 0$$

A) [10 points] Use the implicit function theorem to calculate $\frac{dx^*}{d\alpha}$.

B) [10 points] Use the envelope theorem to estimate the maximized value of the objective function when $\alpha = -3.9$.

**Problem 4 (True or False) [32 points]:**
Answer whether each of the following true or false. Each part is worth 4 points.

T) If the statement is *true*, while a rigorous proof is not essential, your credit will increase with the thoroughness of your answer. You don't have to reprove results that were covered in class, but if you cite a theorem taught in class, try to make clear which theorem it is taht you are citing.

F) If the statement is *false*, your credit will increase the more you are able to: (i) give a counter-example (this will be useful and easy to construct for some parts *but not all parts*); (ii) write down/sketch a statement that is true, and as closely related as posible to the statement you've declared to be false; (iii) explain why your counterexample to the statement is *not* a counter example to your correct statement. Some false statements have more than one thing wrong with them; for full credit identify both wrong things.

A small amount of credit may be given for a one letter answer. In the first two parts, the NPP problem is $\max f(\mathbf{x})$ s.t. $g(\mathbf{x}) \leq \mathbf{b}$, where $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$.

A) Necessary conditions for $\mathbf{x}$ to be a solution to the NPP problem are that the constraint qualification is satisfied at $\mathbf{x}$ and the KKT conditions are satisfied at $\mathbf{x}$.

B) If $\mathrm{Hg}(\cdot) = 0$, then a necessary condition for $\mathbf{x}$ to be a solution to the NPP problem is that the KKT conditions are satisfied at $\mathbf{x}$.

C) Let $f(\mathbf{x}) = \mathbf{a}+\mathbf{b}\cdot\mathbf{x}+\mathbf{c}\cdot\mathbf{x}^2$, where $\mathbf{x}, \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$. Given $\mathbf{dx}$, the larger is the $\mathbf{c}$ vector relative to the $\mathbf{b}$ vector, the more nonlinear is the differential, so that approximating $f(\mathbf{x} + \mathbf{dx})$ by a first order Taylor expansion of $f$ around $\mathbf{x}$ is more likely to result in a sign error.

D) Let $f : \mathbb{R}^n \to \mathbb{R}$ be an $k$'th order polynomial, and fix $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. One can obtain the value of $f$ at $\mathbf{y}$ by constructing a $k-1$-th order Taylor expansion of $f$ at $\mathbf{x}$.

E) A twice continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is quasi-convex at $\mathbf{x}$ but not strictly quasi-convex at $\mathbf{x}$ if $\mathbf{dx}'\mathrm{H}\mathbf{f}(\mathbf{x})\mathbf{dx} \geq 0$, for all $\mathbf{dx} \neq 0$ such that $\bigtriangledown f(\mathbf{x})\mathbf{dx} = 0$.

F) A twice continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is quasi-concave if its Hessian is globally negative definite.

G) A necessary and sufficient condition for $f : \mathbb{R}^n \to \mathbb{R}$ to attain a strict local maximum at $\mathbf{x} \in \mathbb{R}^n$ is that $\bigtriangledown f(\mathbf{x}) = 0$ and $\mathbf{dx}'\mathrm{H}\mathbf{f}(\mathbf{x})\mathbf{dx} < 0$, for all $\mathbf{dx} \neq 0$.

H) Let $f : \mathbb{R}^n \to \mathbb{R}$ be thrice continuously differentiable and fix $\mathbf{x} \in \mathbb{R}^n$. For any $\mathbf{dx} \in \mathbb{R}^n$ such that $\mathbf{dx}'\mathrm{H}\mathbf{f}(\mathbf{x})\mathbf{dx} \neq 0$, $\exists\epsilon > 0$ s.t. if $||\mathbf{dx}|| < \epsilon$,

$$| \bigtriangledown f(\mathbf{x})\mathbf{dx} + 0.5\mathbf{dx}'\mathrm{H}\mathbf{f}(\mathbf{x})\mathbf{dx} > |\text{remainder term}|.$$