Final Exam - Answer key

Problem 1 (Calculus) [24 points]:

Compute the partial derivatives of the following functions and then determine whether or not they are differentiable at (0,0). Justify your answers.

A) [12 points] $f(x,y) = (x^2 y)^{\frac{1}{3}}$.

Ans: The function is not differentiable at (0,0). To see this, we show that one of the directional derivatives cannot be written as a linear combination of the partials. The partials are given by

$$f_x(x,y) = \lim_{t\to 0} \frac{f((x,y)+(t,0)) - f(x,y)}{t}$$
$$= \lim_{t\to 0} \frac{(t^2(0))^{\frac{1}{3}} - 0}{t} = 0$$
$$f_y(x,y) = \lim_{t\to 0} \frac{f((x,y)+(0,t)) - f(x,y)}{t}$$
$$= \lim_{t\to 0} \frac{(0^2(t))^{\frac{1}{3}} - 0}{t} = 0$$

The directional derivative in direction (a,a), a$\neq 0$. is given by

$$\lim_{|k|\to\infty} \frac{f(0+\frac{a}{k}, 0+\frac{a}{k}) - f(0,0)}{||(a,a)||/k}$$
$$= \lim_{|k|\to\infty} \frac{\left(\left(\frac{a}{k}\right)^2 \frac{a}{k}\right)^{\frac{1}{3}} - 0}{|a|/\sqrt{2}k}$$
$$= \lim_{|k|\to\infty} \frac{\frac{a}{k}}{|a|/\sqrt{2}k}$$
$$= \frac{sgn(a)}{\sqrt{2}}$$

As this is non-zero, it cannot be written as a linear combination of the partials, so the function is not differentiable.

B) [12 points]
$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right), & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

Ans: This function is differentiable at (0,0). To see this, let's calculate the directional derivatives in direction (h_1, h_2) at (0,0).

$$\lim_{|k|\to\infty} \frac{f(0+\frac{h_1}{k}, 0+\frac{h_2}{k}) - f(0,0)}{||(h_1,h_2)||/k}$$

$$= \lim_{|k|\to\infty} \frac{\left[\left(\frac{h_1}{k}\right)^2 + \left(\frac{h_2}{k}\right)^2\right] \sin\left(\frac{1}{\sqrt{\left(\frac{h_1}{k}\right)^2+\left(\frac{h_2}{k}\right)^2}}\right) - 0}{||(h_1, h_2)||/k}$$

$$= \lim_{|k|\to\infty} \frac{\left[\frac{1}{k}[h_1^2 + h_2^2\right] \sin\left(\frac{1}{\frac{1}{|k|}\sqrt{h_1^2+h_2^2}}\right)}{||(h_1, h_2)||}$$

$$= 0$$

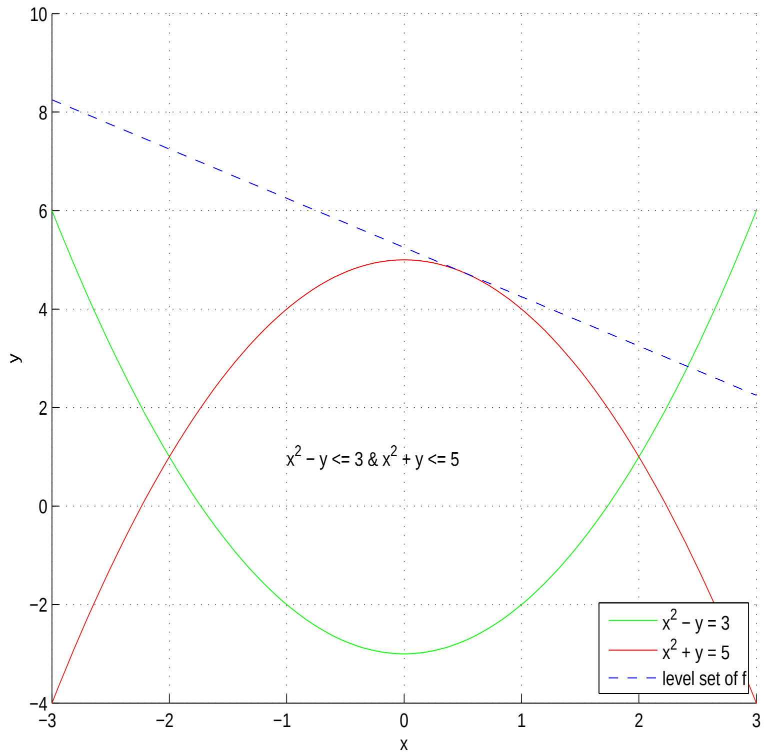
Thus, the directional derivative in every direction is zero, so every directional derivative can be written as a linear combination of the partials, which are also zero.

Problem 2 (Constrained Optimization) [24 points]:

Consider the following constrained maximization problem.

$$\max x + y$$
$$\text{subject to } g1 : x^2 - y \leq 3$$
$$g2 : x^2 + y \leq 5$$

A) [4 points] Sketch the constraint set and draw a level set of the utility function.



Ans:

B) [4 points] Does a solution to the problem exist? Explain.

Ans: Yes, a solution exists. The constraints form a closed and bounded set (therefore compact), and the objective function is continuous. Thus, the function obtains a maximum by the extreme value theorem.

C) [6 points] Write down the Lagrange and the KKT conditions.

Ans: The Lagrangian is given by

$$L = x + y + \lambda_1(3 - x^2 + y) + \lambda_2(5 - x^2 - y)$$

(1) $\frac{\partial L}{\partial x} = 1 - 2\lambda_1 x - 2\lambda_2 x = 0$
(2) $\frac{\partial L}{\partial y} = 1 + \lambda_1 - \lambda_2 = 0$
(3) $\frac{\partial L}{\partial \lambda_1} = 3 - x^2 + y \geq 0$ $\quad (6)\lambda_1(3 - x^2 + y) = 0$ $\quad (9)\lambda_1 \geq 0$
(4) $\frac{\partial L}{\partial \lambda_2} = 5 - x^2 - y \geq 0$ $\quad (7)\lambda_2(5 - x^2 - y) = 0$ $\quad (10)\lambda_2 \geq 0$

D) [10 points] Find the solution to the problem. Show your work for all cases for maximum credit.

Ans: Case 1: Interior solution, $\lambda_1 = 0, \lambda_2 = 0$
$\nabla f = (1,1)$
The gradient is never vanishing. No interior solution.
Case 2: $\lambda_1 > 0, \lambda_2 > 0$

There are two points where both constraints are satisfied with equality.

$$g_1 \Rightarrow x^2 = y + 3$$

$$g_2 \Rightarrow x^2 = -y + 5$$

$$y + 3 = -y + 5 \Rightarrow y = 1$$

Case 2a: (2,1)
(1) $\Rightarrow 1 - 2\lambda_1(2) - 2\lambda_2(2) = 0$
(2) $\Rightarrow 1 + \lambda_1 - \lambda_2 = 0$
$\Rightarrow \lambda_2 = 1 + \lambda_1$
$\Rightarrow 1 - 4\lambda_1 - 4(1 + \lambda_1) = 0$
$\Rightarrow -3 - 8\lambda_1 = 0$
$\Rightarrow \lambda_1 = -\frac{3}{8} < 0$ Contradiction.
Case 2b: (-2,1)
(1) $\Rightarrow 1 - 2\lambda_1(-2) - 2\lambda_2(-2) = 0$
(2) $\Rightarrow 1 + \lambda_1 - \lambda_2 = 0$
$\Rightarrow \lambda_2 = 1 + \lambda_1$
$\Rightarrow 1 + 4\lambda_1 + 4(1 + \lambda_1) = 0$
$\Rightarrow 5 + 8\lambda_1 = 0$
$\Rightarrow \lambda_1 = -\frac{5}{8} < 0$
Contradiction.
We check the constraint qualification for Case 2. The gradients of the constraints satisfied with equality are
$\nabla g = \begin{pmatrix} 2x & -1 \\ 2x & 1 \end{pmatrix}$ which are not collinear at (-2,1) (nor at (2,1)). Case 3: $\lambda_1 > 0, \lambda_2 = 0$

The gradients of the constraints satisfied with equality are
$\nabla g = \begin{pmatrix} 2x & -1 \end{pmatrix}$ This is full rank, so the constraint qualification is satisfied.

$$(2) \Rightarrow 1 + \lambda_1 = 0$$

$$\lambda_1 < 0$$

Contradiction. Case 4: $\lambda_2 > 0, \lambda_1 = 0$

The gradients of the constraints satisfied with equality are

$\nabla g = \begin{pmatrix} 2x & 1 \end{pmatrix}$ so again the constraint qualification is satisfied.

$$(2) \Rightarrow 1 - \lambda_2 = 0$$

$$\lambda_2 = 1$$

$$(1) 1 - 2\lambda_2 x = 0$$

$$x = \frac{1}{2}$$

$$(7) \Rightarrow y = 5 - \left(\frac{1}{2}\right)^2 = 4.75$$

Potential solution. (0.5, 4.75), f(0.5,4.75)=5.25
Comparing the two potential solutions, the final solution occurs at (0.5,4.75) with a maximum of 5.25.

Problem 3 (Comparative Statics) [20 points]:
Consider the following constrained maximization problem.

$$\max x + \alpha \text{ subject to } x^2 + \alpha \leq 0$$

A) [10 points] Use the implicit function theorem to calculate $\frac{dx^*}{d\alpha}$.

Ans: Note that the constraint is always binding, so $\lambda > 0$ at all solutions. The Lagrangian of the function is

$$L = x + \alpha + \lambda(-x^2 - \alpha)$$

x^* is defined implicitly by the level sets

$$L_x = 1 - 2\lambda x = 0$$

$$L_\lambda = -x^2 - \alpha = 0$$

$\begin{pmatrix} L_{xx} & L_{x\lambda} \\ L_{\lambda x} & L_{\lambda\lambda} \end{pmatrix} = \begin{pmatrix} -2\lambda & -2x \\ -2x & 0 \end{pmatrix}$ The determinant is given by $-4x^2$. $\begin{pmatrix} L_{x\alpha} \\ L_{\lambda\alpha} \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$
$\frac{dx^*}{d\alpha} = -\det \begin{pmatrix} 0 & -2x \\ -1 & 0 \end{pmatrix} / -4x^2 = \frac{-(-2x)}{-4x^2} = -\frac{1}{2x}$

B) [10 points] Use the envelope theorem to estimate the maximized value of the objective function when $\alpha = -3.9$.

Ans: The maximized value can be approximated by

$$f(x^*(\alpha)) + \frac{df(x^*(\alpha))}{d\alpha} d\alpha$$

when $\alpha = -4$ and $d\alpha = 0.1$. $x^*(\alpha = -4) = 2$, $\lambda^*(\alpha = -4) = \frac{1}{4}$ from the first order conditions. From the envelope theorem,

$$\frac{df(x^*(\alpha))}{d\alpha} = \frac{\partial f(x^*(\alpha))}{\partial \alpha} + \lambda^*(\alpha)\frac{\partial g(x^*(\alpha))}{\partial \alpha}$$

$$= 1 - \lambda^*(\alpha) = \frac{3}{4}$$

The maximized objective function is then

$$f(x^*(\alpha)) + \frac{df(x^*(\alpha))}{d\alpha} d\alpha = 2 + (-4) + \frac{3}{4}\left(\frac{1}{10}\right) = -\frac{80}{40} + \frac{3}{40} = -\frac{77}{40} = -1.925$$

Problem 4 (True or False) [32 points]:
Answer whether each of the following true or false. Each part is worth 4 points.

T) If the statement is *true*, while a rigorous proof is not essential, your credit will increase with the thoroughness of your answer. You don't have to reprove results that were covered in class, but if you cite a theorem taught in class, try to make clear which theorem it is taht you are citing.

F) If the statement is *false*, your credit will increase the more you are able to: (i) give a counter-example (this will be useful and easy to construct for some parts *but not all parts*); (ii) write down/sketch a statement that is true, and as closely related as posible to the statement you've declared to be false; (iii) explain why your counterexample to the statement is *not* a counter example to your correct statement. Some false statements have more than one thing wrong with them; for full credit identify both wrong things.

A small amount of credit may be given for a one letter answer. In the first two parts, the NPP problem is $\max f(\mathbf{x})$ s.t. $g(\mathbf{x}) \leq \mathbf{b}$, where $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$.

A) Necessary conditions for $\mathbf{x}$ to be a solution to the NPP problem are that the constraint qualification is satisfied at $\mathbf{x}$ and the KKT conditions are satisfied at $\mathbf{x}$.

Ans: *False*: A solution may exist for the NPP problem even though both the constraint qualification and the KKT conditions are violated at $\mathbf{x}$. A counter-example was provided in class (beamerNPP1-12.pdf). The following would be a correct statement: if the constraint qualification is satisfied at $\mathbf{x}$, a necessary conditions for $\mathbf{x}$ to be a solution to the NPP problem is that the KKT conditions are satisfied at $\mathbf{x}$.

B) If $\mathrm{Hg}(\cdot) = 0$, then a necessary condition for $\mathbf{x}$ to be a solution to the NPP problem is that the KKT conditions are satisfied at $\mathbf{x}$.

Ans: *True*: The role of the constraint qualification is to ensure that when the non-linear constraint set is replaced by its linearization (i.e., the tangent planes to the constraints), the resulting linear constraint set is locally a very close approximation to the original constraint set. If $\mathrm{Hg}(\cdot) = 0$, then the constraints are all affine functions, and hence the linearized version of the constraint set is just the constraint set itself.

C) Let $f(\mathbf{x}) = \mathbf{a} + \mathbf{b} \cdot \mathbf{x} + \mathbf{c} \cdot \mathbf{x}^2$, where $\mathbf{x}, \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$. Given $\mathbf{dx}$, the larger is the $\mathbf{c}$ vector relative to the $\mathbf{b}$ vector, the more nonlinear is the differential, so that approximating $f(\mathbf{x} + \mathbf{dx})$ by a first order Taylor expansion of f around $\mathbf{x}$ is more likely to result in a sign error.

Ans: *False*: The differential is a linear function. A correct statement would be: "For given $\mathbf{dx}$, the larger is the $\mathbf{c}$ vector relative to the $\mathbf{b}$ vector, approximating $f(\mathbf{x} + \mathbf{dx})$ by a first order Taylor expansion of f around $\mathbf{x}$ is more likely to result in a sign error because the remainder term is more likely to be larger in absolute value than the value of the differential when evaluated at $\mathbf{dx}$."

D) Let $f : \mathbb{R}^n \to \mathbb{R}$ be an k'th order polynomial, and fix $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. One can obtain the value of f at $\mathbf{y}$ by constructing a $k-1$-th order Taylor expansion of f at $\mathbf{x}$.

Ans: *True*: $f(\mathbf{y})$ can be written as the sum of the terms in a $k-1$-th order Taylor expansion of f around $\mathbf{x}$ plus a remainder term. By the Taylor-Lagrange theorem, the remainder term of a $k-1$-th order Taylor expansion involves the k'th order derivative of f, which is constant for a k'th order polynomial. Hence the remainder term can be computed exactly.

E) A twice continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is quasi-convex at $\mathbf{x}$ but not strictly quasi-convex at $\mathbf{x}$ if $\mathbf{dx}'\mathsf{Hf}(\mathbf{x})\mathbf{dx} \geq 0$, for all $\mathbf{dx} \neq 0$ such that $\bigtriangledown f(\mathbf{x})\mathbf{dx} = 0$.

Ans: *False*: This statement is wrong on two counts. First, quasi-convexity is a global concept, so that a property of the Hessian of f evaluated at a single-point, provides no information about whether a function is quasi-convex. Second, if the "$\mathbf{dx}$" in the statement about the Hessian were true for all $\mathbf{x}$ instead of a specific one, the statement would still be false: you can't conclude from "for all $\mathbf{x}$, $\mathbf{dx}'\mathsf{Hf}(\mathbf{x})\mathbf{dx} \leq 0$, for all $\mathbf{dx} \neq 0$ such that $\bigtriangledown f(\mathbf{x}) = 0$." that the function is *not* strictly quasi-concave. A true statement would be: A twice continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is quasi-convex if for all $\mathbf{x} \in \mathbb{R}^n$, if $\mathbf{dx}'\mathsf{Hf}(\mathbf{x})\mathbf{dx} \geq 0$, for all $\mathbf{dx} \neq 0$ such that $\bigtriangledown f(\mathbf{x})\mathbf{dx} = 0$.

F) A twice continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is quasi-concave if its Hessian is globally negative definite.

Ans: *True*: Global negative definiteness is a sufficient condition for global negative semidefiniteness on the subspace orthogonal to the gradient, which is the standard sufficiency condition for quasi-concavity.

G) A necessary and sufficient condition for $f : \mathbb{R}^n \to \mathbb{R}$ to attain a strict local maximum at $\mathbf{x} \in \mathbb{R}^n$ is that $\bigtriangledown f(\mathbf{x}) = 0$ and $\mathbf{dx}'\mathsf{Hf}(\mathbf{x})\mathbf{dx} < 0$, for all $\mathbf{dx} \neq 0$.

Ans: *False*: The condition is sufficient but not necessary. A counter example is $f(x) = -x^4$. This attains a strict *global* maximum at $\mathbf{0}$, but $\mathsf{Hf}(\mathbf{0})$ is zero, hence the condition "$\mathbf{dx}'\mathsf{Hf}(\mathbf{x})\mathbf{dx} < 0$, for all $\mathbf{dx} \neq 0$" fails at zero.

H) Let $f : \mathbb{R}^n \to \mathbb{R}$ be thrice continuously differentiable and fix $\mathbf{x} \in \mathbb{R}^n$. For any $\mathbf{dx} \in \mathbb{R}^n$ such that $\mathbf{dx}'\mathsf{Hf}(\mathbf{x})\mathbf{dx} \neq 0$, $\exists \epsilon > 0$ s.t. if $||\mathbf{dx}|| < \epsilon$,

$$| \bigtriangledown f(\mathbf{x})\mathbf{dx} + 0.5\mathbf{dx}'\mathsf{Hf}(\mathbf{x})\mathbf{dx} > |\text{remainder term}|.$$

Ans: *True*: The condition is stronger than necessary. It says that you have to choose the direction $\mathbf{dx}$ before you choose ϵ. In fact you can choose an $\epsilon > 0$ that works for all directions $\mathbf{dx}$. Thus the statement is a strictly stronger version of the theorem given on slide #11 of beamerCalculus3.