Fall2011 **ARE211**

FINAL EXAM
WEDNESDAY, DEC 15, 2011

This is the final exam for ARE211. As announced earlier, this is an open-book exam. However, use of computers, calculators, Iphones, Ipads, cell phones, Blackberries and other non-human aids with on-off switches is forbidden.

Read all questions carefully before starting the test.

Allocate your 180 minutes in this exam wisely. The exam has 100 points, so aim for an *average* of about 1.75 minute per point. However, some questions & parts are distinctly easier than others. Make sure that you first do all the easy parts, before you move onto the hard parts. Always bear in mind that if you leave a part-question completely blank, you cannot conceivably get any marks for that part. On the other hand, to give a 100% complete answer to all of the questions would almost certainly take you longer than the time available. (Writing the answer key took me quite a while.) In some problems, it may be efficient to simply write down the correct answer (if you know what it is) then come back and justify the answer later.

The questions are designed so that, to some extent, even if you cannot answer some parts, you will still be able to answer later parts. Even if you are unable to show a result, you are allowed to use the result in subsequent parts of the question. So don't hesitate to leave a part out. *if you get stuck on something, move on and come back later.* You don't have to answer questions and parts of questions in the order that they appear on the exam, *provided that you clearly indicate the question/part-question you are answering.* Finally, *always* keep in mind the famous maxim KISS (keep it simple, stupid).

If a question doesn't specify a metric, assume the usual Euclidean metric.

Problem 1 (Real Analysis) [24 points]:

A) [6 points] Let A, B be subsets of a set X. Prove

$$A \cap B = \emptyset \qquad \Leftrightarrow \qquad A \subseteq X \backslash B$$

B) [6 points] Use the open cover definition of compactness to prove that an arbitrary finite subset of $\mathbb{R}$ is compact in $\mathbb{R}$. Your answer should hold for all metrics on $\mathbb{R}$.

C) [6 points] Let (X, d) be a metric space. For $A \subseteq X$, define the distance between $x \in X$ and A to be

$$r(x, A) = \inf_{a \in A} d(x, a).$$

Prove that A is closed if and only if $x \in A$ for any $x \in X$ such that $r(x, A) = 0$.

D) [6 points] Define the function $f : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ where $f(x, \alpha) = x^\alpha - x$. Define the correspondence $\Psi(\alpha) : \mathbb{R}_+ \twoheadrightarrow \mathbb{R}$ as

$$\Psi(\alpha) = \{x \in \mathbb{R} : \frac{\partial f(x; \alpha)}{\partial x} \neq 0\}.$$

On what subset of $\mathbb{R}_+$ is $\Psi(\cdot)$ compact-valued? Prove.

Problem 2 (Linear Algebra) [12 points]:
For parts A and B answer the following questions: Is S a vector space. (Possible answers are yes, no or maybe. If not, prove this with an example. If it is, (a) what is its dimension? (b) Give a basis for S. (c) Pick arbitrarily an element of S that is *not part of your basis* and show how it can be written as a linear combination of your basis set. If your answer is maybe, indicate under what conditions the answer is "yes", and under these conditions answer (a)-(c) above.

A) [6 points] $S = \Big\{ y : \mathbb{N} \to \mathbb{R} : \exists \mathbf{x} \in \mathbb{R}^3 \text{ s.t. } y = \{x_1, x_2, x_3, x_1, x_2, x_3, x_1, x_2, x_3 ...\} \Big\}$.

B) [6 points] Fix $\mathbf{x} \in \mathbb{R}^3$ and let $S = \Big\{ y : \mathbb{N} \to \mathbb{R} : y = \{x_1, x_2, x_3, x_1, x_2, x_3, x_1, x_2, x_3 ...\} \Big\}$.

Problem 3 (Calculus I) [15 points]:
Let

$$g(x, y, z) = \begin{bmatrix} e^{2x} - \frac{z}{y^2} \\ xyz - y^4 \end{bmatrix}$$

A) [5 points] Compute the Jacobian of $g(\cdot)$, $Jg(x, y, z)$

B) [5 points] Compute the directional derivative at $g(0, 1, 2)$ in the direction $h = (0, 0, -2)$.

C) [5 points] Compute a first order approximation of $g(1, 1, 3)$ from the point $g(0, 1, 2)$

Problem 4 (Calculus II) [14 points]:
Let $f(x, y, z) = e^{2x} - \frac{z}{y^2}$. It will make calculations easier to note that this is the first element of function $g(\cdot)$ from the last problem.

A) [7 points] Compute a second order approximation of $f(1, 1, 3)$ from the point $f(0, 1, 2)$.

B) [7 points] For $dx = (1, 0, 1)$, does there exist a neighborhood around $(0, 1, 2)$ such that the signs of the first order Taylor Approximation about $(0, 1, 2)$ agrees with the sign of $f((0, 1, 2) + dx) - f((0, 1, 2))$? What about for the sign of the second order Taylor Approximation?

Problem 5 (Constrained Optimization) [25 points]:
Consider the NPP

$$\max_{x_1,x_2} -(x_1-c_1)^2-(x_2-c_2)^2 \qquad \text{s.t. } (x_1+1)^2+x_2^2 \leq 4,\ x_1 \geq 0,\ x_2 \geq 0$$

where $c_1, c_2 \in \mathbb{R}$. For convenience, denote the objective function $f(x_1,x_2) = -(x_1-c_1)^2-(x_2-c_2)^2$.

A) [5 points] Does a solution to this problem exist? Explain why. (Don't use intuition for this part, be formal.)

B) [5 points] Construct graphs in (x_1, x_2) space to solve this problem graphically for (a) $c_1 = c_2 = -0.5$ (b) $c_1 = -4, c_2 = 0$. Make sure each that your graphs includes: at least two level sets of the objective; the boundary of the constraint set; the solution; gradient vectors whereever you feel they are appropriate.

C) [5 points] Are the KKT conditions necessary for a solution of this problem? Prove or disprove. Hint: You can answer this rigorously, which will take quite a bit of time, or fairly quickly, possibly with reference to the answer to the previous part. You can get almost all of the maximum available marks (maybe all but one) if you can indicate successfully that you know what you are doing. Depending on how much you've mastered the material, this if could be a BIG if.

D) [5 points] For the two answers you gave in part B, what can you say about the lagrangian of the constraint at the solution value.

E) [5 points] Identify a c vector such that at the solution to the problem with this c vector , the constraint is satisfied with equality is but not binding.

Problem 6 (Comparative Statics) [10 points]:

Let $S(p) = 1 + p^2$, and $D(p,t) = 5 - (p + t)$, where S is supply, D is demand, p denotes price and $t \geq 0$ is a tax paid by consumers. An equilibrium for this system is a price $p \geq 0$such that $S(p) = D(p,t)$.

A) [5 points] Write down a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that a necessary condition for (p,t) to solve this system is that (p,t) belongs to the level set of f corresponding to zero. Is this condition also sufficient for p to be an equilibrium price given t? If not, add an additional condition so that your function and additional condition are jointly necessary and sufficient for an equilibrium to exist.

B) [5 points] Set $t = 2$, and write down a function that represents *to a first order approximation* how p changes with t. Hint: When $t = 2$ the equilibrium value of p is 1. For what positive values of dt does your first order approximation correctly predict the sign of the equlibrium value of p? (Note: the question asks about the sign of p *and not* about the sign of dp.)