FINAL EXAM - ANSWER KEY

Problem 1 (Real Analysis) [24 points]:

A) [6 points] Let A, B be subsets of a set X. Prove

$$A \cap B = \emptyset \qquad \Leftrightarrow \qquad A \subseteq X \backslash B$$

Ans: $\Rightarrow$: If $A = \emptyset$, then we are done, since the empty set is a subset of any set. Then suppose $A \neq \emptyset$. Consider any $x \in A$. By $A \cap B = \emptyset$, then $x \notin B$, or $x \in X \backslash B$. Thus, $A \subseteq X \backslash B$.
$\Leftarrow$: If $A = \emptyset$, then again we are done since $\emptyset \cap B = \emptyset$ for any set B. Consider any $x \in A$. Since $A \subseteq X \backslash B$, then $x \in X \backslash B$, or $x \in X$ and $x \notin B$. Thus, if $x \in A$ then $x \notin B$. so $A \cap B = \emptyset$.

B) [6 points] Use the open cover definition of compactness to prove that an arbitrary finite subset of $\mathbb{R}$ is compact in $\mathbb{R}$. Your answer should hold for all metrics on $\mathbb{R}$.

Ans: Consider an arbitrary finite subset $\{x_1, ..., x_n\}$ of $\mathbb{R}$. Let $\mathcal{U} = \{U_\lambda, \lambda \in \Lambda\}$ be an arbitrary open cover of $\{x_1, ..., x_n\}$. For $i = 1, ...n$, there exists $\lambda_i \in \Lambda$ such that $x_i \in U_{\lambda_i}$. $(U_{\lambda_i})_{i=1}^n$ is a finite cover of x and $(U_{\lambda_i})_{i=1}^n \subset \mathcal{U}$. Thus, an arbitrary open cover of $\{x_1, ..., x_n\}$ has a finite subcover.

C) [6 points] Let (X, d) be a metric space. For $A \subseteq X$, define the distance between $x \in X$ and A to be

$$r(x, A) = \inf_{a \in A} d(x, a).$$

Prove that A is closed if and only if $x \in A$ for any $x \in X$ such that $r(x, A) = 0$.

Ans: $\Rightarrow$: Consider $x \in X$ such that $r(x, A) = 0$. Then $\inf_{a \in A} d(x, a) = 0$.
If A is finite, then we must have $x \in A$, otherwise if $x \notin A$, we would have $\inf_{a \in A} d(x, a) > 0$.
If A is infinite, then $\inf_{a \in A} d(x, a) = 0$ implies for any $\epsilon > 0$, there exists an element $a \in A$ such that $d(x, a) < \epsilon$. Thus, we could construct a sequence of elements from the set A that converges to x, which implies $x \in A$ since A closed.
$\Leftarrow$: Let $x \in A$ for any $x \in X$ such that $r(x, A) = 0$. We prove that A contains all of its accumulation points and is therefore closed. Let x be an accumulation point of A (if A has no accumulation points, then we are done). Then for any $\epsilon > 0$, there exists a point $a \in A$ such that $a \in B(x, \epsilon | X)$. Thus, $\inf_{a \in A} d(x, a) = 0$, so $a \in A$ by our assumption.

D) [6 points] Define the function $f : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ where $f(x, \alpha) = x^\alpha - x$. Define the correspondence $\Psi(\alpha) : \mathbb{R}_+ \twoheadrightarrow \mathbb{R}$ as

$$\Psi(\alpha) = \{x \in \mathbb{R} : \frac{\partial f(x;\alpha)}{\partial x} \neq 0\}.$$

On what subset of $\mathbb{R}_+$ is $\Psi(\cdot)$ compact-valued? Prove.

Ans: $\Psi(\alpha)$ is compact valued only for $\alpha = 1$.
Consider $\alpha = 1$. Then $\frac{\partial f(x;\alpha)}{\partial x} = 0$ for any x, so $\Psi(1) = \emptyset$ which is compact. Now consider $\alpha \neq 1$. For $\alpha = 0$, then $\frac{\partial f(x;0)}{\partial x} = -1$ for any x, so $\Psi(0) = \mathbb{R}$, which is not compact. For $\alpha \in \mathbb{R}_{++}$, $\alpha \neq 1$ we prove $\Psi(\alpha)$ is not compact by showing for any $\alpha \in \mathbb{N}$, $\frac{\partial f(x,\alpha)}{\partial x} \neq 0$ for all but one value of x, which implies $\Psi(\alpha)$ is not closed and therefore not compact. We have $\frac{\partial f(x;\alpha)}{\partial x} = \alpha x^{\alpha-1} - 1$ which

is nonzero iff $x \neq \alpha^{1-\alpha}$. But $\mathbb{R}_+ \backslash \{\alpha^{1-\alpha}\}$ is not a closed set, hence $\Psi(\alpha)$ is not compact valued.

Problem 2 (Linear Algebra) [12 points]:
For parts A and B answer the following questions: Is S a vector space. (Possible answers are yes, no or maybe. If not, prove this with an example. If it is, (a) what is its dimension? (b) Give a basis for S. (c) Pick arbitrarily an element of S that is *not part of your basis* and show how it can be written as a linear combination of your basis set. If your answer is maybe, indicate under what conditions the answer is "yes", and under these conditions answer (a)-(c) above.

A) [6 points] $S = \Big\{ y : \mathbb{N} \to \mathbb{R} : \exists \mathbf{x} \in \mathbb{R}^3 \text{ s.t. } y = \{x_1, x_2, x_3, x_1, x_2, x_3, x_1, x_2, x_3 ...\} \Big\}.$

Ans: S is a vector space. (a) Its dimension is 3. (b) A basis set is $y^1 = \{1, 0, 0, 1, 0, 0, 1, 0, 0, ...\}$, $y^2 = \{0, 1, 0, 0, 1, 0, 0, 1, 0, ...\}$, and $y^3 = \{0, 0, 1, 0, 0, 1, 0, 0, 1, ...\}$. (c) Let $\mathbf{z}$ be a nonzero element of $\mathbb{R}^3$ such that $z_i \neq 0$, for $i = 1, ..., 3$ and consider the sequence $y = \{z_1, z_2, z_3, z_1, z_2, z_3, z_1, z_2, z_3 ...\}$. Clearly $y \in S$ and since all elements are nonzero, it is not an element of the basis set. Moreover, $y = \sum_{i=1}^{3} z_i y_i$.

B) [6 points] Fix $\mathbf{x} \in \mathbb{R}^3$ and let $S = \Big\{ y : \mathbb{N} \to \mathbb{R} : y = \{x_1, x_2, x_3, x_1, x_2, x_3, x_1, x_2, x_3 ...\} \Big\}.$

Ans: Maybe. It is a vector space for $\mathbf{x} = 0$, for all other $\mathbf{x}$'s it is not a vector space. If $\mathbf{x} = 0$, then it's a zero-dimensional vector space consisting of the single element $y = \{0, 0, ...\}$. If $\mathbf{x} \neq 0$, then let $y = \{\mathbf{x}, \mathbf{x}, \mathbf{x}, ...\}$. Clearly $2y \notin S$, verifying that S is not a vector space.

Problem 3 (Calculus I) [15 points]:
Let

$$g(x,y,z) = \begin{bmatrix} e^{2x} - \frac{z}{y^2} \\ xyz - y^4 \end{bmatrix}$$

A) [5 points] Compute the Jacobian of $g(\cdot)$, $Jg(x,y,z)$

Ans:

$$Jg(x,y,z) = \begin{bmatrix} 2e^{2x} & \frac{2z}{y^3} & -\frac{1}{y^2} \\ yz & xz - 4y^3 & xy \end{bmatrix}$$

B) [5 points] Compute the directional derivative at $g(0,1,2)$ in the direction $h = (0,0,-2)$.

Ans: The directional derivative is $g_h(x,y,z = Jg(x,y,z) \cdot \frac{h}{||h||}$. Note that $||h|| = 2$. Hence

$$\begin{aligned} g_h(0,1,2) &= Jg(x,y,z) \cdot \frac{h}{||h||} \\ &= \begin{bmatrix} 2e^{2x} & \frac{2z}{y^3} & -\frac{1}{y^2} \\ yz & xz - 4y^3 & xy \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ \text{-2/2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{y^2} \\ -xy \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned}$$

C) [5 points] Compute a first order approximation of $g(1,1,3)$ from the point $g(0,1,2)$

Ans:

$$\begin{aligned} g(1,1,3) &\approx g(0,1,2) + Jg(0,1,2)dx \qquad \text{where } dx = (1,0,1) \\ &= \begin{bmatrix} \text{-1} \\ \text{-1} \end{bmatrix} + \begin{bmatrix} 2e^{2x} & \frac{2z}{y^3} & -\frac{1}{y^2} \\ yz & xz - 4y^3 & xy \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \text{-1} \\ \text{-1} \end{bmatrix} + \begin{bmatrix} 2 & 4 & -1 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \text{-1} \\ \text{-1} \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned}$$

Problem 4 (Calculus II) [14 points]:
Let $f(x,y,z) = e^{2x} - \frac{z}{y^2}$. It will make calculations easier to note that this is the first element of function $g(\cdot)$ from the last problem.

A) [7 points] Compute a second order approximation of $f(1,1,3)$ from the point $f(0,1,2)$.

Ans:

$$Hf(x,y,z) = \begin{bmatrix} 4e^{2x} & 0 & 0 \\ 0 & -\frac{6z}{y^4} & \frac{2}{y^3} \\ 0 & \frac{2}{y^3} & 0 \end{bmatrix}$$

$$Hf(0,1,2) = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -12 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

$$f(1,1,3) \approx f(0,1,2) + \triangledown f(0,1,2)dx + \frac{1}{2}dx' Hf(0,1,2)dx \qquad \text{where } dx = (1,0,1)$$

$$= 0 + \frac{1}{2}[1,0,1] \begin{bmatrix} 4 & 0 & 0 \\ 0 & -12 & 2 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{2}[1,0,1] \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$$

$$= 2$$

B) [7 points] For $dx = (1,0,1)$, does there exist a neighborhood around $(0,1,2)$ such that the signs of the first order Taylor Approximation about $(0,1,2)$ agrees with the sign of $f((0,1,2)+dx) - f((0,1,2))$? What about for the sign of the second order Taylor Approximation?

Ans: Yes for both. First note that f is thrice continuously differentiable.
For the second order Taylor approximation, we know that the conditions for Local Taylor are met since $dx'Hf(0,1,2)dx = 2 \neq 0$ as shown in the last part. Therefore, we know the second order Taylor Approximation of the change has magnitude greater than the magnitude of the remainder term, so the sign of the change is correct.
Similarly, for the first order Taylor Approximation, we know $\triangledown f(0,1,2)dx = 1 \neq 0$. Thus, we can use Local Taylor again and we know the sign of the change is correct

Problem 5 (Constrained Optimization) [25 points]:
Consider the NPP

$$\max_{x_1,x_2} -(x_1-c_1)^2-(x_2-c_2)^2 \qquad \text{s.t. } (x_1+1)^2+x_2^2 \le 4,\ x_1 \ge 0,\ x_2 \ge 0$$

where $c_1, c_2 \in \mathbb{R}$. For convenience, denote the objective function $f(x_1,x_2) = -(x_1-c_1)^2-(x_2-c_2)^2$.

A) [5 points] Does a solution to this problem exist? Explain why. (Don't use intuition for this part, be formal.)

Ans: Yes. The constraints form a closed and bounded, thus compact set. The objective function is continuous. Thus, by the EVT, a solution exists.

B) [5 points] Construct graphs in (x_1,x_2) space to solve this problem graphically for (a) $c_1 = c_2 = -0.5$ (b) $c_1 = -4, c_2 = 0$. Make sure each that your graphs includes: at least two level sets of the objective; the boundary of the constraint set; the solution; gradient vectors whereever you feel they are appropriate.

Ans: See figure 1.

C) [5 points] Are the KKT conditions necessary for a solution of this problem? Prove or disprove. Hint: You can answer this rigorously, which will take quite a bit of time, or fairly quickly, possibly with reference to the answer to the previous part. You can get almost all of the maximum available marks (maybe all but one) if you can indicate successfully that you know what you are doing. Depending on how much you've mastered the material, this if could be a BIG if.

Ans: The constraints clearly form a compact set. Thus, we only need to check the constraint qualification. Let

$$g^1(x_1,x_2) = -(x_1+1)^2 - x_2^2 - 4$$
$$g^2(x_1,x_2) = -x_1$$
$$g^3(x_1,x_2) = -x_2$$

Note the Jacobian of the constraints is given by

$$\begin{bmatrix} \frac{\partial g^1}{\partial x_1} & \frac{\partial g^1}{\partial x_2} \\ \frac{\partial g^2}{\partial x_1} & \frac{\partial g^2}{\partial x_2} \\ \frac{\partial g^3}{\partial x_1} & \frac{\partial g^3}{\partial x_2} \end{bmatrix} = -\begin{bmatrix} 2(x_1+1) & 2x_2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Case 1: $x_1 = 0 = x_2$. Then only g^2, g^3 hold with equality, and the Jacobian of constraints that hold with equality is

$$-\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

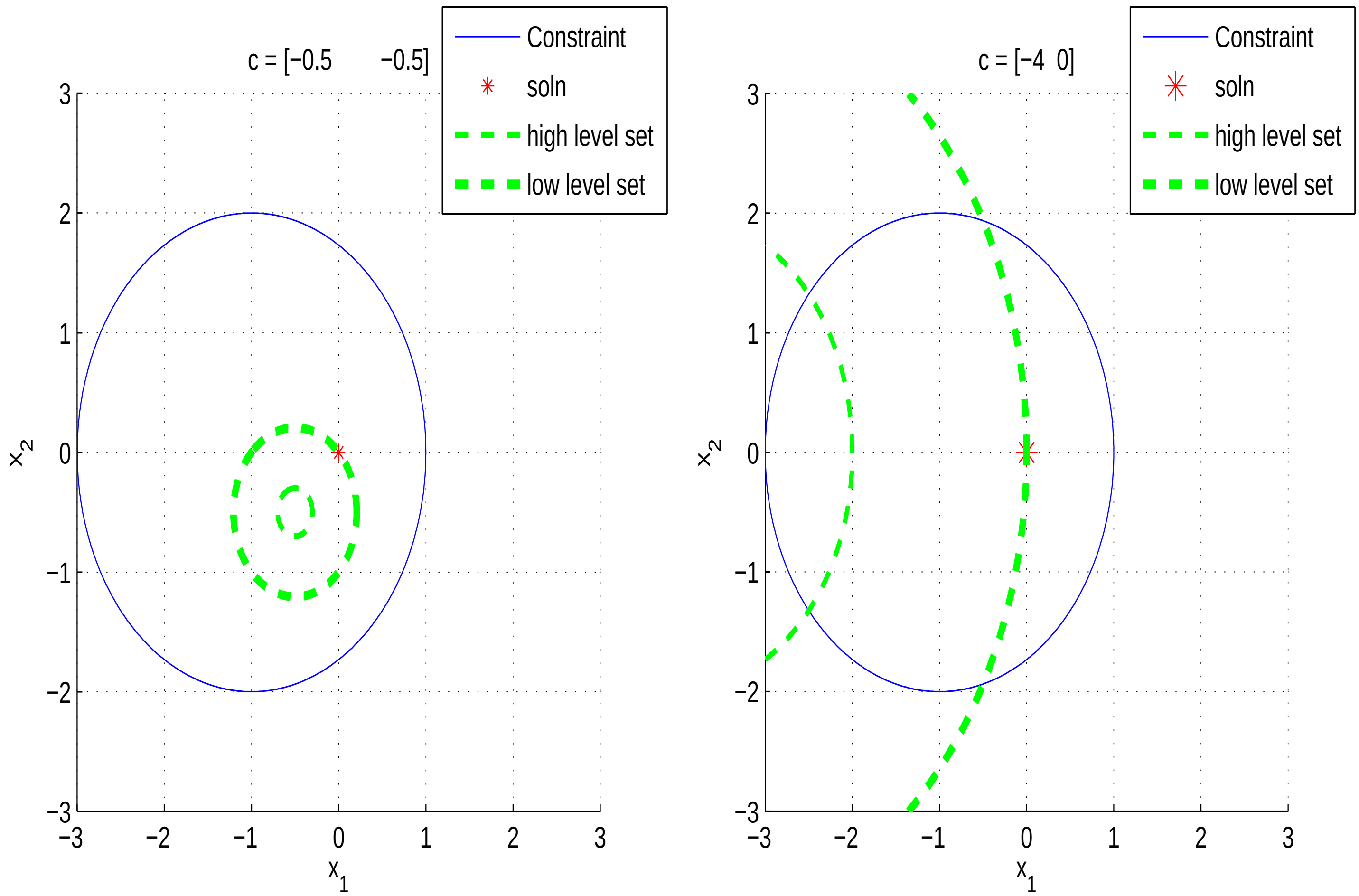

FIGURE 1. KKT Question

which is clearly full rank.
Case 2: $x_1 > 0$, $x_2 = 0$. Then g^1 and g^3 both could hold with equality, and the relevant Jacobian is

$$-\begin{bmatrix} 2(x_1+1) & 0 \\ 0 & 1 \end{bmatrix},$$

which is full rank since $2(x_1+1) > 0$ by $x_1 > 0$. It is also possible that only g^3 holds with equality, and clearly the Jacobian of g^3 forms a linearly independent set.
Case 3: $x_1 = 0$, $x_2 > 0$. This is similar to Case 2. We know the gradient of g^1 and g^2 are linearly independent in this case since the former is $-[2, 2x_2]$ which cannot be a scalar multiple of $-[1, 0]$ since $x_2 > 0$. If g^2 is the only constraint that holds with equality, the gradient of g^2 is linearly independent, so the CQ holds.
Case 4: $x_1 > 0$, $x_2 > 0$. Then only g^1 can hold with equality. If it does, then the gradient of g^1 cannot be the 0 vector since $2(x_1+1) > 0$ and $2x_2 > 0$ by our $x_1,\ x_2 > 0$.
Thus, for any possible value of x_1, x_2, the CQ holds.
Thus, the KKT conditions are necessary for a solution.

D) [5 points] For the two answers you gave in part B, what can you say about the lagrangian of the constraint at the solution value.

Ans: For (a): the lagrangian is 0 on the first constraint and positive on the second constraints; for (b): the lagrangian is 0 for the first and third constraint and positive for the second constraint.

E) [5 points] Identify a c vector such that at the solution to the problem with this c vector , the constraint is satisfied with equality is but not binding.

Ans: $c = (1, 0)$; For this case, the solution is $(1, 0)$, which is the bliss point of the objective function. The solution lies on the boundary of the constraint; if the constraint were slackened, the optimum would remain the same.

Problem 6 (Comparative Statics) [10 points]:

Let $S(p) = 1 + p^2$, and $D(p,t) = 5 - (p + t)$, where S is supply, D is demand, p denotes price and $t \geq 0$ is a tax paid by consumers. An equilibrium for this system is a price $p \geq 0$such that $S(p) = D(p,t)$.

A) [5 points] Write down a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that a necessary condition for (p,t) to solve this system is that (p,t) belongs to the level set of f corresponding to zero. Is this condition also sufficient for p to be an equilibrium price given t? If not, add an additional condition so that your function and additional condition are jointly necessary and sufficient for an equilibrium to exist.

Ans: $f(p,t) = 4 - (p + t) - p^2$. This condition alone is necessary but not sufficient: p could be negative if t is too large. An additional condition is that $t \leq 4$. The combined conditions $t \leq 4$ and $f(p,t) = 0$ are necessary and sufficient for p to be an equilibrium given t.

B) [5 points] Set $t = 2$, and write down a function that represents *to a first order approximation* how p changes with t. Hint: When $t = 2$ the equilibrium value of p is 1. For what positive values of dt does your first order approximation correctly predict the sign of the equlibrium value of p? (Note: the question asks about the sign of p *and not* about the sign of dp.)

Ans: Applying the implicit function theorem to the level set of $f(p,t)$ corresponding to zero, we obtain $\frac{dp(t)}{dt} = -1/(1 + 2p(t))$. So $\left.\frac{dp(t)}{dt}\right|_{t=2} = -1/(1 + 2) = -1/3$. Therefore $p(2+dt) \approx p(2) - dt/3 = 1 - dt/3$. For $dt \leq 2$, the sign of the approximation is the same as the sign of the true value of p, i.e., positive, but for $dt \in (2,3)$ the approximation yields a positive value of price, while the true value is negative. (For $dt > 3$ both approximation and the actual equilibrium equation predict a negative price. Both are wrong.)