Final Exam

Wednesday, Dec 15, 2010

This is the final exam for ARE211. As announced earlier, this is an open-book exam. However, use of computers, calculators, Iphones, Ipads, cell phones, Blackberries and other non-human aids with on-off switches is forbidden.

Read all questions carefully before starting the test.

Allocate your 180 minutes in this exam wisely. The exam has 180 points, so aim for an *average* of 1 minute per point. However, some questions & parts are distinctly easier than others. Make sure that you first do all the easy parts, before you move onto the hard parts. Always bear in mind that if you leave a part-question completely blank, you cannot conceivably get any marks for that part. On the other hand, to give a 100% complete answer to all of the questions would almost certainly take you longer than the time available. (Writing the answer key took me quite a while.) In some problems, it may be efficient to simply write down the correct answer (if you know what it is) then come back and justify the answer later.

The questions are designed so that, to some extent, even if you cannot answer some parts, you will still be able to answer later parts. Even if you are unable to show a result, you are allowed to use the result in subsequent parts of the question. So don't hesitate to leave a part out. *if you get stuck on something, move on and come back later.* You don't have to answer questions and parts of questions in the order that they appear on the exam, *provided that you clearly indicate the question/part-question you are answering.* Finally, *always* keep in mind the famous maxim KISS (keep it simple, stupid).

Problem 1 (Real Analysis) [36 points]:

Answer whether each of the following statements is true or false. If true, prove your answer; if false provide a counterexample. If you have trouble giving a formal proof, or constructing a formal counterexample, a helpful picture will usually earn you partial credit.

A) [6 points] Let (x_n) be a sequence in $\mathbb{R}$ such that for all n, $x_n > 0$. If 0 is the greatest lower bound (GLB) for the set $\{x_n : n \in \mathbb{N}\}$, then the sequence contains a strictly decreasing subsequence. (Hint: think about GLB's);

B) [6 points] If $\{x_n\}$ is a Cauchy sequence and $f(\cdot)$ a continuous function, then $\{f(x_n)\}$ is a Cauchy sequence.

C) [6 points] The product of two homogenous functions is a homogenous function. (Definition: A function $f : \mathbb{R}^n_+ \to \mathbb{R}$ is *homogenous* if for some $k > 0$, all $\mathbf{x} \in \mathbb{R}^n_+$ and all $\alpha > 0$, $f(\alpha\mathbf{x}) = \alpha^k f(\mathbf{x})$.

D) [6 points] A quasiconcave function defined on an open convex set $S \subset \mathbb{R}$ does not obtain a strict global minimum on S.

E) [6 points] If $f_1(\cdot), f_2(\cdot), \ldots, f_n(\cdot)$ are quasiconcave functions, and $g(\cdot)$ is a monotone function, then $h(\cdot) = g(f_1(\cdot) + f_2(\cdot) + \ldots + f_n(\cdot))$ is a quasiconcave function.

F) [6 points] The correspondence $\Psi : \mathbb{R}_+ \twoheadrightarrow \mathbb{R}_+$ defined by $\Psi(x) = \left\{y : \; 0 \leq y < \frac{1}{x} \text{ if } x > 0; \; \emptyset \text{ if } x = 0\right\}$is upper hemicontinuous.

Problem 2 (Linear Algebra) [36 points]:

For $i = 1, 2$, let $\mathbf{v}^i$ and λ^i denote a unit-length eigenvector and corresponding eigenvalue for invertible 2×2 matrix A. Assume that $\lambda^1 \neq \lambda^2$.
It will probably help to do part C) before parts A) and B).

A) [6 points] For $i = 1, 2$, let $\mathbf{u}^i$ and ψ^i denote, respectively, a unit eigenvector and eigenvalue of A^{-1}. Find $\mathbf{u}^i$ and ψ^i.

B) [6 points] For $i = 1, 2$, let $\mathbf{w}^i$ and ϕ^i denote, respectively, a unit eigenvector and eigenvalue of A^k. Find $\mathbf{w}^i$ and ϕ^i.

C) [6 points] Explain the intuition behind your results using the interpretation of A as a mapping of the unit circle to $\mathbb{R}^2$.

Given a vector space V, indicate whether each statement is true or false. If true, prove your claim. If false, give a counterexample

D) [6 points] If the vectors $\mathbf{u}^1$, $\mathbf{u}^2$ and $\mathbf{u}^3$ span V, then $\dim V = 3$.

E) [6 points] If the vectors $\mathbf{u}^1$, $\mathbf{u}^2$ and $\mathbf{u}^3$ are a minimal spanning set for V, then $\dim V = 3$.

F) [6 points] If the vectors $\mathbf{u}^1$, $\mathbf{u}^2$ and $\mathbf{u}^3$ are a minimal spanning set for V, then $\mathbf{u}^1$, $\mathbf{u}^2$, $\mathbf{u}^3$ are linearly independent.

Problem 3 (Calculus) [36 points]:

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f = \sqrt{x_1} x_2^2$

A) [5 points] Write down the gradient and the hessian of f

B) [5 points] Write down the directional derivative $f_h(\cdot)$ where $h = (3, 4)$

C) [5 points] Let $\mathbf{x} = (1, 2)$. Write down the differential of $\bigtriangledown f$ at $\mathbf{x}$.

D) [5 points] Let $\mathbf{x} = (1, 2)$. Use the differential to approximate the value of $\bigtriangledown f$ at $(2, 5)$

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that for some points x and x' in $\mathbb{R}$, $f(x') > f(x)$. (Hint: the difference in the differentiability conditions in the next two parts is a big HINT.)

E) [8 points] If f is *once* continuously differentiable, use Global Taylor to show that there exists an x'' between x and x' such that the slope of f at x'' is equal to the slope of the line segment joining $(x, f(x))$ and $(x', f(x'))$.

F) [8 points] If f is *twice* continuously differentiable, show that if $f'(x) = 0$, then there exists a point y between x and x', such that $f''(y) > 0$.

Problem 4 (Constrained Optimization) [36 points]:

You are given the following maximization problem:

$$\max_{x,y} 2x + 3y \quad s.t. \quad \sqrt{x} + \sqrt{y} \leq 5, x, y \geq 0$$

A) [8 points] Find the values for x, y an the lagrangians that satisfy the Kuhn Tucker conditions. If you prefer, solve the first order conditions of the Lagrangian.

B) [8 points] Find the solution to the maximization problem.

C) [6 points] Your answers to the first two parts should not be the same. Explain why they are not.

Let $f : X \to \mathbb{R}$ be defined by $f(\mathbf{x}) = x_1 x_2$, for some convex set $X \subset \mathbb{R}_2^+$,

D) [8 points] Let $X = \{\mathbf{x} \in \mathbb{R}_+^2 : \mathbf{x} \neq 0\}$. Using the definition of negative definiteness "subject to constraint" or "on a subspace," prove that f is strictly quasi-concave.

E) [6 points] Is f pseudo-concave? Prove your answer.

Problem 5 (Comparative Statics) [36 points]:

A competitive market equilibrium is described by the following two equations:

Profit maximization: $p\,(nq) - \frac{\partial c(q;w)}{\partial q} = 0$

Zero profit: $qp\,(nq) - c\,(q;w) = 0$

where $p(\cdot)$ denotes price, n the number of firms—ignore that firms come in integers, i.e., treat n as a real number.—q denotes quantity produced by each individual firm, $c(\cdot,\cdot)$ costs and w input costs. Assume that $p' < 0$ and that $c = \alpha + wq^2$, $\alpha > 0$.

A) [14 points] Find the derivative of n and q with respect to changes in w.

B) [14 points] Does the quantity produced by each firm increase or decrease if w increases? What about the number of firms?

C) [8 points] The total quantity produced by all firms is $Q = nq$. What is the approximate change in Q when w increases by 0.1 units? What is the sign of this change?