Final Exam - Answer key

Problem 1 (Real Analysis) [36 points]:

Answer whether each of the following statements is true or false. If true, prove your answer; if false provide a counterexample. If you have trouble giving a formal proof, or constructing a formal counterexample, a helpful picture will usually earn you partial credit.

A) [6 points] Let (x_n) be a sequence in $\mathbb{R}$ such that for all n, $x_n > 0$. If 0 is the greatest lower bound (GLB) for the set $\{x_n : n \in \mathbb{N}\}$, then the sequence contains a strictly decreasing subsequence. (Hint: think about GLB's);

Ans: **TRUE**:
Let $y_1 = x_1$. Now assume that for $k \in \mathbb{N}$, we've chosen $(y_1, ..., y_k)$ such that for some strictly increasing function $\tau^k : (1, ..., k) \to \mathbb{N}$, $y_k = x_{\tau^k(k)}$, thus $(y_1, ..., y_k)$ preserves the order of (x_n). Moreover, assume that for all $k' \leq k$, $y_{k'-1} > y_{k'}$. Both conditions are satisfied vacuously for $k = 1$. Now by assumption, $y_k > 0$. Since 0 is a GLB for the image of the sequence, there exists $n > \tau^k(k)$ such that $x_n \in (0, y_k)$. Set y_{k+1} equal to this x_n and define $\tau^{k+1} : (1, ..., k+1) \to \mathbb{N}$ by, $\tau^{k+1}(k') = \begin{cases} \tau^k(k') & \text{if } k' \leq k \\ n & \text{if } k' = k+1 \end{cases}$. Since $n > \tau^k(k)$, τ^{k+1} is a strictly increasing function. Moreover, for all $k' \leq k+1$, $y_{k'-1} > y_{k'}$. Hence, by induction we have constructed a strictly decreasing subsequence of (x_n).

B) [6 points] If $\{x_n\}$ is a Cauchy sequence and $f(\cdot)$ a continuous function, then $\{f(x_n)\}$ is a Cauchy sequence.

Ans: **FALSE**:
Let $f : \mathbb{R}^{++} \to [-1, 1]$ be defined by $f(x) = sin(1/x)$; The sequence $x = \left(\frac{2}{\pi(1+2n)}\right)_{n\in\mathbb{N}}$ converges to zero, and so is Cauchy. Yet $f(x) = \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases}$. Thus $\{f(x_n)\}$ is not a Cauchy sequence.

C) [6 points] The product of two homogenous functions is a homogenous function.
(Definition: A function $f : \mathbb{R}^n_+ \to \mathbb{R}$ is *homogenous* if for some $k > 0$, all $\mathbf{x} \in \mathbb{R}^n_+$ and all $\alpha > 0$, $f(\alpha\mathbf{x}) = \alpha^k f(\mathbf{x})$.

Ans: **TRUE**:
Let $h(\mathbf{x}) = f(\mathbf{x})g(\mathbf{x})$ where f and g are both homogeneous. Then there exists k^f, k^g for all $\mathbf{x} \in \mathbb{R}^n_+$ and all $\alpha > 0$, $f(\alpha\mathbf{x}) = \alpha^{k^f} f(\mathbf{x})$ and $g(\alpha\mathbf{x}) = \alpha^{k^g} g(\mathbf{x})$. Hence

$$h(\alpha\mathbf{x}) = f(\alpha\mathbf{x})g(\alpha\mathbf{x}) = \alpha^{k^f} f(\mathbf{x})\alpha^{k^g} g(\mathbf{x}) = \alpha^{k^f+k^g} h(\mathbf{x})$$

D) [6 points] A quasiconcave function defined on an open convex set $S \subset \mathbb{R}$ does not obtain a strict global minimum on S.

Ans: **TRUE**:
Suppose f attains a strict global mininum at $s \in S$. Since S is open, s is an interior point of S. It follows that there exists $\epsilon > 0$, such that $s + \epsilon$ and $s - \epsilon$ both belong to S. Let $\alpha = \min(f(s-\epsilon), f(s+\epsilon))$. Since f attains a strict global mininum at $s \in S$, $f(s) < \alpha$. Both $s+\epsilon$ and $s-\epsilon$ belong to the upper contour set of f corresponding to α, but s does not. Hence the upper contour set of f corresponding to α is not a convex set, so that f cannot be quasi-concave.

E) [6 points] If $f_1(\cdot), f_2(\cdot), \ldots, f_n(\cdot)$ are quasiconcave functions, and $g(\cdot)$ is a monotone function, then $h(\cdot) = g(f_1(\cdot) + f_2(\cdot) + \ldots + f_n(\cdot))$ is a quasiconcave function.

Ans: **FALSE**:
Let $n = 2$ and $g(x) = x$. Clearly g is a monotone function. Let $f_1(x) = \begin{cases} x & \text{if } x \in [0,1] \\ 2-x & \text{if } x \in [1,2] \\ 0 & \text{otherwise} \end{cases}$ and let $f_2(x) = \begin{cases} x-2 & \text{if } x \in [2,3] \\ 4-x & \text{if } x \in [3,4] \\ 0 & \text{otherwise} \end{cases}$. Clearly $h(x) = \begin{cases} x & \text{if } x \in [0,1] \\ 2-x & \text{if } x \in [1,2] \\ x-2 & \text{if } x \in [2,3] \\ 4-x & \text{if } x \in [3,4] \\ 0 & \text{otherwise} \end{cases}$. Thus, $f_1(x) + f_2(x)$ is not quasi-concave. Moreover, $h(x) = g(f_1(x) + f_2(x)) = f_1(x) + f_2(x)$, so that h is not a quasi-concave function.

F) [6 points] The correspondence $\Psi : \mathbb{R}_+ \twoheadrightarrow \mathbb{R}_+$ defined by $\Psi(x) = \left\{y : 0 \leq y < \frac{1}{x} \text{ if } x > 0; \emptyset \text{ if } x = 0\right\}$is upper hemicontinuous.

Ans: **FALSE**:
Fix $\epsilon > 0$ arbitrarily. For all $x \in B(x, \epsilon) \cap \mathbb{R}_{++}$, $\Psi(x) \subset [0, 1/\epsilon)$. Hence the upper inverse image $\bar{\Psi}^{-1}\left([0, \frac{1}{2\epsilon})\right) \cap [0, \epsilon) = \{0\}$. Hence $\bar{\Psi}^{-1}\left([0, \frac{1}{2\epsilon})\right)$ is not open. Hence Ψ is not u.h.c.

Problem 2 (Linear Algebra) [36 points]:

For $i = 1, 2$, let $\mathbf{v}^i$ and λ^i denote a unit-length eigenvector and corresponding eigenvalue for invertible 2×2 matrix A. Assume that $\lambda^1 \neq \lambda^2$.
It will probably help to do part C) before parts A) and B).

A) [6 points] For $i = 1, 2$, let $\mathbf{u}^i$ and ψ^i denote, respectively, a unit eigenvector and eigenvalue of A^{-1}. Find $\mathbf{u}^i$ and ψ^i.

Ans: $\mathbf{u}^i = \mathbf{v}^i$, $\psi^i = 1/\lambda^i$.

B) [6 points] For $i = 1, 2$, let $\mathbf{w}^i$ and ϕ^i denote, respectively, a unit eigenvector and eigenvalue of A^k. Find $\mathbf{w}^i$ and ϕ^i.

Ans: $\mathbf{w}^i = \mathbf{v}^i$, $\phi^i = k\lambda^i$.

C) [6 points] Explain the intuition behind your results using the interpretation of A as a mapping of the unit circle to $\mathbb{R}^2$.

Ans:

Given a vector space V, indicate whether each statement is true or false. If true, prove your claim. If false, give a counterexample

D) [6 points] If the vectors $\mathbf{u}^1$, $\mathbf{u}^2$ and $\mathbf{u}^3$ span V, then $\dim V = 3$.

Ans: False: let $\mathbf{u}^i = i$, for $i = 1, 2, 3$. These three vectors span $\mathbb{R}$, whose dimension is 1.

E) [6 points] If the vectors $\mathbf{u}^1$, $\mathbf{u}^2$ and $\mathbf{u}^3$ are a minimal spanning set for V, then $\dim V = 3$.

Ans: False: let $\mathbf{u}^1 = (1, 0, 0)$, $\mathbf{u}^2 = (0, 1, 1)$, $\mathbf{u}^3 = (0, 1, -1)$. These vectors space $\{\mathbf{v} \in \mathbb{R}^3 : \mathbf{v}_3 = 0\}$, which is a 2 dimensional subspace of $\mathbb{R}^3$.

F) [6 points] If the vectors $\mathbf{u}^1$, $\mathbf{u}^2$ and $\mathbf{u}^3$ are a minimal spanning set for V, then $\mathbf{u}^1$, $\mathbf{u}^2$, $\mathbf{u}^3$ are linearly independent.

Ans: True: suppose $\mathbf{u}^3 = t_1\mathbf{u}^1 + t_2\mathbf{u}^2$, so that the set is not linear independent. Now consider $\mathbf{v} \in V$; if $\mathbf{v} = \sum_{i=1}^{3} r_i\mathbf{u}^i$, then $\mathbf{v} = \sum_{i=1}^{2} r_i\mathbf{u}^i + r_3(t_1\mathbf{u}^1 + t_2\mathbf{u}^2)$, i.e., $\mathbf{v}$ can be written as a linear combination of $\mathbf{u}^1$ and $\mathbf{u}^2$.

Problem 3 (Calculus) [36 points]:

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f = \sqrt{x_1}x_2^2$

A) [5 points] Write down the gradient and the hessian of f

Ans: $\bigtriangledown f(\mathbf{x}) = \begin{bmatrix} \frac{x_2^2}{2\sqrt{x_1}} & 2x_2\sqrt{x_1} \end{bmatrix}$. $Hf(\mathbf{x}) = \begin{bmatrix} \frac{x_2^2}{4\sqrt{x_1^3}} & \frac{x_2}{\sqrt{x_1}} \\ \frac{x_2}{\sqrt{x_1}} & 2\sqrt{x_1} \end{bmatrix}$

B) [5 points] Write down the directional derivative $f_h(\cdot)$ where $h = (3, 4)$

Ans: $f_h = \frac{3x_2^2}{10\sqrt{x_1}} + \frac{8x_2\sqrt{x_1}}{5} = \frac{3x_2^2+16x_2x_1}{10\sqrt{x_1}}$

C) [5 points] Let $\mathbf{x} = (1, 2)$. Write down the differential of $\bigtriangledown f$ at $\mathbf{x}$.

Ans: $d \bigtriangledown f^{\mathbf{x}}(\mathbf{dx}) = Hf(\mathbf{x})\mathbf{dx} = \begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix}$.

D) [5 points] Let $\mathbf{x} = (1, 2)$. Use the differential to approximate the value of $\bigtriangledown f$ at $(2, 5)$

Ans: Let $\mathbf{dx} = (2, 5) - (1, 2) = (1, 3)$

$$\begin{aligned} \bigtriangledown f(2,4) &\approx \bigtriangledown f(\mathbf{x}) + d \bigtriangledown f^{\mathbf{x}}(\mathbf{dx}) \\ &= [2, 4] + \begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}. \quad = \quad [7, 12] \end{aligned}$$

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that for some points x and x' in $\mathbb{R}$, $f(x') > f(x)$. (Hint: the difference in the differentiability conditions in the next two parts is a big HINT.)

E) [8 points] If f is *once* continuously differentiable, use Global Taylor to show that there exists an x'' between x and x' such that the slope of f at x'' is equal to the slope of the line segment joining $(x, f(x))$ and $(x', f(x'))$.

Ans: Let $dx = x' - x$. By Global Taylor, there exists $\lambda \in [0, 1]$, such that $f(x') - f(x) = f'(x + \lambda dx)dx$. Hence $\frac{f(x')-f(x)}{dx} = f'(x + \lambda dx)$. But $\frac{f(x')-f(x)}{dx}$ is the slope of the line segment joining $(x, f(x))$ and $(x', f(x'))$. Hence, for $x'' = x + \lambda dx$, $f'(x'')$ equals the slope of the line segment.

F) [8 points] If f is *twice* continuously differentiable, show that if $f'(x) = 0$, then there exists a point y between x and x', such that $f''(y) > 0$.

Ans: By Global Taylor, there exists $\lambda \in [0, 1]$, such that $f(x') - f(x) = f'(x)dx + 0.5f''(x + \lambda dx)dx^2$. Since $f(x') - f(x) > 0$ and $f'(x)dx = 0$ it follows that $f''(x + \lambda dx)dx^2 > 0$. Since $dx^2 > 0$, it must be the case that $f''(x + \lambda dx) > 0$. Let $y = x + \lambda dx$. We've shown that $f''(y) > 0$.

Problem 4 (Constrained Optimization) [36 points]:

You are given the following maximization problem:

$$\max_{x,y} 2x + 3y \quad s.t. \quad \sqrt{x} + \sqrt{y} \leq 5, x, y \geq 0$$

A) [8 points] Find the values for x, y an the lagrangians that satisfy the Kuhn Tucker conditions. If you prefer, solve the first order conditions of the Lagrangian.

Ans:

$$L(x, y, \lambda) = 2x + 3y + \lambda_0(5 - \sqrt{x} - \sqrt{y}) + \lambda_x x + \lambda_y y$$

The first order conditions are

$$\begin{aligned}
\frac{\partial L}{\partial x} &= 2 - \frac{\lambda_0}{2\sqrt{x}} + \lambda_x = 0 \\
\frac{\partial L}{\partial y} &= 3 - \frac{\lambda_0}{2\sqrt{y}} + \lambda_y = 0 \\
\frac{\partial L}{\partial \lambda_0} &= (5 - \sqrt{x} - \sqrt{y}) \geq 0 \\
\frac{\partial L}{\partial \lambda_x} &= x \geq 0 \\
\frac{\partial L}{\partial \lambda_y} &= y \geq 0 \\
\text{for } i \in \{0, x, y\}, \qquad \lambda_i \frac{\partial L}{\partial \lambda_i} &\geq 0
\end{aligned}$$

The objective function is strictly increasing in both x and y so the solution to this problem must satisfy $\sqrt{x} + \sqrt{y} = 5$.

(a) If $x, y > 0$ then $\lambda_x = \lambda_y = 0$ in which case, $4\sqrt{x} = 6\sqrt{y}$, i.e., $\sqrt{y} = 2/3\sqrt{x}$ or $5\sqrt{x}/3 = 5$ or $x = 9$ hence $y = 4$. If $\lambda_0 = 12$, then $\frac{dL}{dx} = 2 - \frac{12}{6} = \frac{dL}{dy} = 3 - \frac{12}{4} = 0$. Thus the Kuhn Tucker conditions are satisfied.

(b) Suppose $x = 0$ so that $y = 25$. By the complementary slackness conditions, $\lambda_x = 0$, so that $\frac{dL}{dx} > 0$.

(c) Suppose $y = 0$ so that $x = 25$. By the complementary slackness conditions, $\lambda_y = 0$, so that $\frac{dL}{dy} > 0$.

Conclude that the only values that solve the problem are $x = 9$, $y = 4$, $\lambda_0 = 12$, $\lambda_x = \lambda_y = 0$. Note for future reference that at these values, the value of the objective function is $18+12 = 30$.

B) [8 points] Find the solution to the maximization problem.

Ans: If $y = 25$, $x = 0$, the value of the objective is 75. If $y = 0$, $x = 25$, the value of the objective is 55. Thus the solution to the maximization problem is $y = 25$, $x = 0$.

C) [6 points] Your answers to the first two parts should not be the same. Explain why they are not.

Ans: The constraint function is not quasi-convex. This means that solving the KT conditions does not guarantee a maximum. In other words, the fact that the Kuhn Tucker conditions are satisfied at $x = 9$, $y = 4$ does *not* mean that these values *necessarily* maximize the value of the objective on the constraint set.

Let $f : X \to \mathbb{R}$ be defined by $f(\mathbf{x}) = x_1 x_2$, for some convex set $X \subset \mathbb{R}^+_2$,

D) [8 points] Let $X = \{\mathbf{x} \in \mathbb{R}^2_+ : \mathbf{x} \neq 0\}$. Using the definition of negative definiteness "subject to constraint" or "on a subspace," prove that f is strictly quasi-concave.

Ans: $\bigtriangledown f(\mathbf{x}) = (x_2, x_1)$; $Hf(\mathbf{x}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Fix $\mathbf{x} \neq 0$, & $\mathbf{dx} \neq 0$ such that $\bigtriangledown f(\mathbf{x})\mathbf{dx} = 0$. Assume without loss of generality that $x_1 \neq 0$. Necessarily, $dx_2 = -\frac{x_2 dx_1}{x_1}$. Hence

$$\begin{bmatrix} dx_1 & -\frac{x_2 dx_1}{x_1} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} dx_1 \\ -\frac{x_2 dx_1}{x_1} \end{bmatrix} = \begin{bmatrix} dx_1 & -\frac{x_2 dx_1}{x_1} \end{bmatrix} \begin{bmatrix} -\frac{x_2 dx_1}{x_1} & dx_1 \end{bmatrix} = -2\frac{x_2 dx_1^2}{x_1} < 0$$

E) [6 points] Is f pseudo-concave? Prove your answer.

Ans: Answer is yes iff X does *not* contain zero. Assume that X doesn't contain zero. We've just proved that in this case, f is strictly quasi-concave, and hence quasi-concave. Also, $\bigtriangledown f(\mathbf{x}) = (x_2, x_1)$ which is non-zero when $\mathbf{x} \neq 0$. Hence f is quasi-concave and the gradient never vanishes, and so is pseudo-concave. On the other hand, if X contains zero, $\bigtriangledown f(0) = 0$, and 0 is not a global maximum of f; hence f fails the test for pseudo-concavity.

Problem 5 (Comparative Statics) [36 points]:

A competitive market equilibrium is described by the following two equations:

Profit maximization: $p(nq) - \frac{\partial c(q;w)}{\partial q} = 0$

Zero profit: $qp(nq) - c(q;w) = 0$

where $p(\cdot)$ denotes price, n the number of firms—ignore that firms come in integers, i.e., treat n as a real number.—q denotes quantity produced by each individual firm, $c(\cdot,\cdot)$ costs and w input costs. Assume that $p' < 0$ and that $c = \alpha + wq^2$, $\alpha > 0$.

A) [14 points] Find the derivative of n and q with respect to changes in w.

Ans: Parts A) and B) are answered together

B) [14 points] Does the quantity produced by each firm increase or decrease if w increases? What about the number of firms?

Ans: The quantity produced by each firm decreases, but the effect on the number of firms cannot be determined. To verify this, we'll write the equation system as

$$f(q,n;w) = \begin{bmatrix} p(nq) - \frac{\partial c(q,w)}{\partial q} \\ qp(nq) - c(q,w) \end{bmatrix} = 0$$

By the implicit function theorem

$$\begin{bmatrix} \frac{dq}{dw} \\ \frac{dn}{dw} \end{bmatrix} = -\begin{bmatrix} \frac{df^1}{dq}, & \frac{df^1}{dn} \\ \frac{df^2}{dq}, & \frac{df^2}{dn} \end{bmatrix}^{-1} \begin{bmatrix} \frac{df^1}{dw} \\ \frac{df^2}{dw} \end{bmatrix}$$

$$= -\,\mathsf{Jf}_{q,w}^{-1} \begin{bmatrix} -\frac{\partial^2 c(q,w)}{\partial q \partial w} \\ -\frac{\partial c(q,w)}{\partial w} \end{bmatrix} = \mathsf{Jf}_{q,w}^{-1} \begin{bmatrix} \frac{\partial^2 c(q,w)}{\partial q \partial w} \\ \frac{\partial c(q,w)}{\partial w} \end{bmatrix}$$

where

$$\mathsf{Jf}_{q,w} = \begin{bmatrix} np'(nq) - \frac{\partial^2 c(q,w)}{\partial q^2}, & qp'(nq) \\ p(nq) + qnp'(nq) - \frac{\partial c(q,w)}{\partial q}, & q^2 p'(nq) \end{bmatrix} = \begin{bmatrix} np'(nq) - \frac{\partial^2 c(q,w)}{\partial q^2}, & qp'(nq) \\ qnp'(nq), & q^2 p'(nq) \end{bmatrix};$$

terms cancel nicely, so that

$$\det(\mathsf{Jf}_{q,n}) = -\frac{\partial^2 c(q,w)}{\partial q^2} q^2 p' > 0$$

Applying Cramer's rule

$$
\begin{aligned}
\frac{dq}{dw} &= \det\left(\begin{bmatrix} \frac{\partial^2 c(q,w)}{\partial q \partial w}, & qp'(nq) \\ \frac{\partial c(q,w)}{\partial w}, & q^2 p'(nq) \end{bmatrix}\right) \Big/ \det\left(\mathsf{Jf}_{q,n}\right) \\
&= qp'(nq) \det\left(\begin{bmatrix} \frac{\partial^2 c(q,w)}{\partial q \partial w}, & 1 \\ \frac{\partial c(q,w)}{\partial w}, & q \end{bmatrix}\right) \Big/ \left(-\frac{\partial^2 c(q,w)}{\partial q^2} q^2 p'\right) \\
&= -\det\left(\begin{bmatrix} \frac{\partial^2 c(q,w)}{\partial q \partial w}, & 1 \\ \frac{\partial c(q,w)}{\partial w}, & q \end{bmatrix}\right) \Big/ \frac{\partial^2 c(q,w)}{\partial q^2} q \\
&= \left(\frac{\partial c(q,w)}{\partial w} \Big/ q - \frac{\partial^2 c(q,w)}{\partial q \partial w}\right) \Big/ \frac{\partial^2 c(q,w)}{\partial q^2}
\end{aligned}
$$

which, if $c(q,w) = wq^2$

$$
= \frac{q - 2q}{2w} = -\frac{q}{2w}
$$

Now for $\frac{dn}{dw}$:

$$
\begin{aligned}
\frac{dn}{dw} &= \det\left(\begin{bmatrix} np'(nq) - \frac{\partial^2 c(q,w)}{\partial q^2}, & \frac{\partial^2 c(q,w)}{\partial q \partial w} \\ qnp'(nq), & \frac{\partial c(q,w)}{\partial w} \end{bmatrix}\right) \Big/ \det\left(\mathsf{Jf}_{q,n}\right) \\
&= n \det\left(\begin{bmatrix} 1 - \frac{\partial^2 c(q,w)}{\partial q^2}/(np'), & \frac{\partial^2 c(q,w)}{\partial q \partial w} \\ q, & \frac{\partial c(q,w)}{\partial w} \end{bmatrix}\right) \Big/ \left(-\frac{\partial^2 c(q,w)}{\partial q^2} q^2\right) \\
&= \frac{n}{q^2}\left[q \frac{\partial^2 c}{\partial q \partial w} \Big/ \frac{\partial^2 c}{\partial q^2} - \frac{\partial c}{\partial w}\left(1 \Big/ \frac{\partial^2 c}{\partial q^2} - 1/(np')\right)\right]
\end{aligned}
$$

which, if $c(q,w) = wq^2$

$$
= \frac{n}{q^2}\left[\frac{q^2}{w} - 2q^2\left(\frac{1}{2wq} - \frac{1}{np'}\right)\right] = \frac{n}{2w} + \frac{1}{p'}
$$

which cannot be signed

C) [8 points] The total quantity produced by all firms is $Q = nq$. What is the approximate change in Q when w increases by 0.1 units? What is the sign of this change?

Ans: The question asks us to evaluate the differential of Q at $dw = 0.1$. Now $dQ = \frac{dQ}{dw}dw$, so that the answer is just $0.1\frac{dQ}{dw}$.

$$\frac{dQ}{dw} = \frac{\partial q}{\partial w}n + \frac{\partial n}{\partial w}q = -\frac{qn}{2w} + \left(\frac{qn}{2w} + \frac{q}{p'}\right) = \frac{q}{p'} < 0$$

Hence the answer is $dQ \approx \frac{q}{10p'} < 0$.