ARE 211 Final Exam, Fall 2009

(100 points)

1. (Total: 10 points) Let x_n be any sequence containing only numbers of the form $1/n$ with $n \in \mathbb{N}$ with no duplicate elements. Prove x_n is a convergent sequence.

2. (Total: 12 points) Consider the correspondence $G: \mathbb{R} \rightrightarrows \mathbb{R}$ defined by $G(x) = (0,\, x^2 + 1)$.

   (a) (2 points) Draw the graph of the correspondence.

   (b) (6 points) This correspondence is either upper hemicontinuous or lower hemicontinuous, but is not both. Which of these properties is not satisfied? Prove your answer. (Hint: Consider $x = 0$.)

   (c) (2 points) Is G a bounded-valued correspondence?

   (d) (2 points) Does G have a closed graph?

3. (Total: 6 points) Consider the following equation:

$$\begin{bmatrix} -6 & 2 \\ 8 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 12 \end{bmatrix}$$

   (a) (3 points) Are the column vectors comprising the 2×2 matrix linearly independent?

   (b) (3 points) If a unique solution exists, then it is the intersection of two sets. Graphically represent these two sets in $\mathbb{R}^2$.

4. (Total: 6 points) Consider figure 1 on the last page. The curve in the figure represents a function $f: (0,7) \to S \in \mathbb{R}$

   (a) (2 points) What is the level set of the function f corresponding to 2?

   (b) (4 points) What are the lower and upper contour sets of the function f corresponding to 2?

5. (Total: 9 points) Consider $f(x, y, z) = xy^2 - x^3yz + z^4$.
   (a) (3 points) Compute $\nabla f(x, y, z)$.
   (b) (3 points) Compute the directional derivative $f(0, 1, 2)$ in the direction $(1, 2, 2)$.
   (c) (3 points) Compute a first order approximation of $f(1, 3, 4)$ using information about f at $(0, 1, 2)$.

6. (Total: 12 points) Each of the following statements can be proven true or false using an example. Indicate whether each statement is true or false and provide an example validating your claim.
   (a) (3 points) A discontinuous function $f : \mathbb{R} \to \mathbb{R}$ can be strictly quasiconcave.
   (b) (3 points) A quasiconcave function can have a non-convex lower contour set for some $x \in \mathbb{R}$.
   (c) (3 points) Strict quasiconvexity guarantees at least one local minimum.
   (d) (3 points) If every upper contour set of a continuous function $f : \mathbb{R} \to \mathbb{R}$ is closed then f is a quasiconcave function.

7. (Total: 12 points) Consider figure 2 on the last page depicts the domain of a continuously differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$. The feasible set is $A = \{x \in \mathbb{R}^2 : \forall j = 1,\ 2,\ 3,\ g_j(x) \leq b_j\} \bigcap \mathbb{R}^2_{++}$. No demonstration is required to support your answer.
   (a) (4 points) Suppose f has a unique global maximum at x*. True or False: It is possible that the following two conditions are simultaneously satisfied.
      i. $\nabla f(x)$ belongs to the non-negative cone formed by $\nabla g_1(x)$ and $\nabla g_2(x)$.
      ii. $\nabla f(z)$ belongs to the non-negative cone formed by $\nabla g_1(z)$ and $\nabla g_2(z)$.
   (b) (4 points) Now suppose we are also told f is quasiconcave. Let $\nabla f(y)$ belong to the non-negative cone formed by $\nabla g_1(y)$ and $\nabla g_3(y)$. True or False: The shadow values of relaxing g_1 and g_3 are both positive.
   (c) (4 points) Drop the supposition from part (a) that f has a unique global maximum at x*, but once again assume that f is quasiconcave. Now suppose that ∇f vanishes in the feasible set. True or False: We know we have an interior solution in $\mathbb{R}^2_{++}$to the maximization of $f(x)$ subject to $g_j(x) \leq b$ for $j = 1,\ 2,\ 3$.

8. (Total: 15 points) Consider the following NPP:

$$\max_{x_1,x_2} f(x_1, x_2) = -(x_1 - 2) - (x_2 - 4)^2 \qquad (1)$$

$$g_1: \; x_2 \leq 4 - x_1 \qquad \text{and} \qquad g_2: \; x_2 \leq 9 - 3x_1$$

$$g_3: \; x_1 \geq 0 \qquad \text{and} \qquad g_4: \; x_2 \geq 0$$

Using graphical methods answer the following:

(a) (3 points) Draw the $x_1 - x_2$ plane with the above constraints, indicate the feasible set.

(b) (3 points) In the same plot draw ∇g_1 and ∇g_2 where the constraint functions associated with g_1 and g_2 intersect (indicate this intersection with a 'b').

(c) (2 points) In the same plot indicate with a 'c' the solution to the UNconstrained maximization of $f(x_1, x_2)$ above. What is ∇f evaluated at this point 'c' ?

(d) (3 points) Draw a level set of $f(x_1, x_2)$ that passes through point 'b' and draw ∇f at point 'b'.

(e) (4 points) Using $\nabla g_1(b)$ and $\nabla g_2(b)$ as guides what does the positioning of $\nabla f(b)$ tell us about whether or not b is a solution to our constrained maximization problem? If b is not a solution then draw another level set of f to identify the solution to our constrained maximization problem.

9. (Total: 8 points) Consider the objective function: $f(x, \alpha) = \ln(x) - \alpha x^2$ with $\alpha > 0$.

(a) (2 points) Derive $x^*(\cdot)$, a function mapping each possible α to the value of x that maximizes $f(\cdot, \alpha)$.

(b) (2 points) Given the above objective function, how will the optimum value of x change with respect to a change in α?

(c) (1 point) Replacing x with $x^*(\alpha)$ into our objective function above we obtain what is commonly called the _________ function.

(d) (1 point) The total derivative of $f(x(\alpha), \alpha)$ w.r.t. to α is given by: $\frac{df}{d\alpha} = \frac{\partial f}{\partial \alpha} + \frac{\partial f}{\partial x(\alpha)} \cdot \frac{dx(\alpha)}{d\alpha}$. Using your work in (a) and (b) replace the right-hand side of this equation with the relevant partial derivatives and derivatives (do not simplify).

(e) (2 points) Using the envelope theorem simplify your answer in part (d).

10. (Total: 10 points) Consider a monopoly who has a constant marginal cost of production m and cannot produce more than Q units due to a capacity constraint (assume this constraint is binding). Also, let there be a per unit tax of τ. How does the equilibrium price and shadow value of capacity change with an increase in τ? (*If you have time, please complete the matrix algebra. Answers that leave the answer in matrix form will not earn more than 6 points*).

Figure 1: $f:\ (0,7) \rightarrow S \in \mathbb{R}$

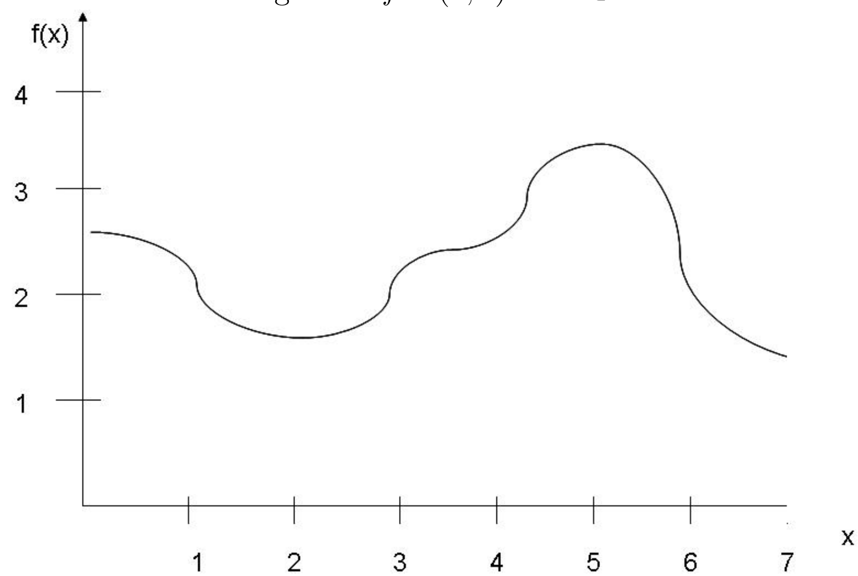

Figure 2: Domain of a continuously differentiable function $f:\ \mathbb{R}^2 \rightarrow \mathbb{R}$