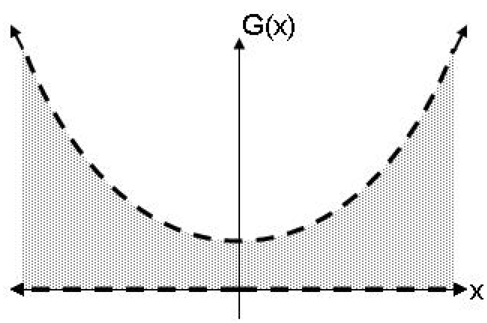
ARE 211 Final Exam, Fall 2009 (100 points)

1. (Total: 10 points) Let x_n be any sequence containing only numbers of the form $1/n$ with $n \in \mathbb{N}$ with no duplicate elements. Prove x_n is a convergent sequence.

   Pick $\epsilon > 0$. We know $\exists N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Define $S = \{n \in \mathbb{N} : x_n \geq \frac{1}{N}\}$. S is a finite set and so has a greatest element, call this $n^* = \max\{S\}$. By construction, $\forall n > n^*$ we have $x_n < \frac{1}{N} < \epsilon$ (otherwise $x_n \geq \frac{1}{N}$ and n would belong to S; hence, $n < n^*$, and we would have a contradiction.). It follows that $\forall n > n^*$ we have $|x_n - 0| < \epsilon$; the sequence x_n converges to zero.

2. (Total: 12 points) Consider the correspondence $G : \mathbb{R} \rightrightarrows \mathbb{R}$ defined by $G(x) = (0,\, x^2 + 1)$.

   (a) (2 points) Draw the graph of the correspondence.



   (b) (6 points) This correspondence is either upper hemicontinuous or lower hemicontinuous, but is not both. Which of these properties is not satisfied? Prove your answer. (Hint: Consider $x = 0$.)

   The correspondence is not upper hemicontinuous. Consider the neighborhood formulation of upper hemicontinuity from the lecture notes. By way of contradiction, suppose G is upper hemicontinuous. Pick $\overline{x} = 0$ and any neighborhood U of $G(0)$, for convenience pick $U = (0, 1)$. Then for $\delta > 0$ there exists a neighborhood V_δ of $\overline{x} = 0$ such that for all $x \in V_\delta$ then $G(x) \subseteq (0, 1)$. Note that $x = \frac{\delta}{2} \in V_\delta$ so that $(\frac{\delta}{2})^2 + 1 \in G(\frac{\delta}{2}) \subseteq (0, 1)$. Hence, $(\frac{\delta}{2})^2 + 1 \in (0, 1)$. Contradiction, and the correspondence is not upper hemicontinuous. Since we are given that G is not both upper and lower hemicontinuous, it follows that the correspondence is not lower hemicontinuous.

   (c) (2 points) Is G a bounded-valued correspondence?

Yes. Pick any x in the domain and $G(x)$ is bounded below by zero and bounded above by $x^2 + 1$.

(d) (2 points) Does G have a closed graph?

No. The graph of G does not contain all of its limit points. For example, all the points lying on the dashed lines in the figure for $2(a)$ are limit points of the graph of G yet do not belong to the graph.

3. (Total: 7 points) Consider figure 1 on the last page. The curve in the figure represents a function $f: (0,7) \to S \in \mathbb{R}$.

(a) (3 points) What is the level set of the function f corresponding to 2?

The level set of f corresponding to 2 contains three points, it is the set $\{x \in (0,7) : f(x) = 2\} = \{1,3,6\}$.

(b) (4 points) What are the lower and upper contour sets of the function f corresponding to 2?

Lower contour set of f is $\{x \in (0,7) : f(x) \leq 2\} = [1,3] \bigcup [6,7)$.
Upper contour set of f is $\{x \in (0,7) : f(x) \geq 2\} = (0,1] \bigcup [3,6]$.

4. (Total: 12 points) Consider $f(x,y,z) = xy^2 - x^3yz + z^4$.

(a) (4 points) Compute $\nabla f(x,y,z)$.
$\nabla f(x,y,z) = [y^2 - 3x^2yz,\ 2xy - x^3z,\ -x^3y + 4z^3]'$

(b) (4 points) Compute the directional derivative $f(0,1,2)$ in the direction $(1,2,2)$.
Recall that $f_h(x) = \nabla f(x) \cdot \frac{h}{\|h\|}$, let $h = (1,2,2)$ and $x = (0,1,2)$.
We easily compute:
$\nabla f(0,1,2) = [1,0,32]'$
$\|h\| = \sqrt{9} = 3$
$f_h(0,1,2) = \frac{1}{3}[1,\ 0,\ 32] \cdot [1,\ 2,\ 2]' = \frac{65}{3}$

(c) (4 points) Compute a first order approximation of $f(1,3,4)$ using information about f at $(0,1,2)$.
Recall that a first order approximation of $f(x)$ using information about f at x_0 is written

as: $f(x) \approx f(x_0) + \nabla f(x_0)dx$ where in this case, $dx = h = (1, 2, 2)$. Plugging in the appropriate numbers we obtain: $f(1, 3, 4) \approx f(0, 1, 2) + \nabla f(0, 1, 2) \cdot [1,\ 2,\ 2]' = 81$.

5. (Total: 12 points) Each of the following statements can be proven true or false using an example. Indicate whether each statement is true or false and provide an example validating your claim.

   (a) (3 points) A discontinuous function $f:\ \mathbb{R} \to \mathbb{R}$ can be strictly quasiconcave.

   True. $f(x) = \begin{cases} x+1 & \text{if } x \geq 0 \\ x-1 & \text{if } x < 0 \end{cases}$

   (b) (3 points) A quasiconcave function can have a non-convex lower contour set for some $x \in \mathbb{R}$.

   True. Consider $f(x) = -x^2$ which is quasiconcave. The lower contour set of the function f corresponding to -1 is the set $\{x : f(x) \leq -1\} = (-\infty, -1] \bigcup [1, \infty)$, which is clearly non-convex.

   (c) (3 points) Strict quasiconvexity guarantees at least one local minimum.

   False. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = x$. This function is strictly quasiconvex, yet has no minimum.

   (d) (3 points) If every upper contour set of a continuous function $f : \mathbb{R} \to \mathbb{R}$ is closed then f is a quasiconcave function.

   False. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = x^2$. Every upper contour set of f is closed, but f is not quasiconcave.

6. (Total: 12 points) Figure 2 on the last page depicts the domain of a continuously differentiable function $f\ :\ \mathbb{R}^2 \to \mathbb{R}$. The feasible set is $A = \{x \in \mathbb{R}^2\ :\ \forall j = 1,\ 2,\ 3,\ g_j(x) \leq b_j\} \bigcap \mathbb{R}^2_{++}$. Briefly support your answer (an example or one to two sentences should be sufficient).

   (a) (4 points) Suppose f has a unique global maximum at x*. True or False: It is possible that the following two conditions are simultaneously satisfied.

      i. $\nabla f(x)$ belongs to the non-negative cone formed by $\nabla g_1(x)$ and $\nabla g_2(x)$.

      ii. $\nabla f(z)$ belongs to the non-negative cone formed by $\nabla g_1(z)$ and $\nabla g_2(z)$.

True. We are given no information about the functional form, f need not be quasiconcave–there are lots of functions such that (i) and (ii) are simultaneously satisfied.

You may have noticed that the original text and figure are not consistent since g_1 does not go through z, and so the non-negative cone formed by $\nabla g_1(z)$ and $\nabla g_2(z)$ is not defined. If you recognized this, then you were also assigned credit.

(b) (4 points) Now suppose we are also told f is quasiconcave and has a unique global maximum at x*. Let $\nabla f(y)$ belong to the non-negative cone formed by $\nabla g_1(y)$ and $\nabla g_3(y)$. True or False: The shadow values of relaxing g_1 and g_3 are both positive.

False. Note that y is not in the feasible set. Since we are given f is quasiconcave it can't be that g_1 and g_3 are both binding.

(c) (4 points) Drop the supposition from part (a) that f has a unique global maximum at x*, but once again assume that f is quasiconcave. Now suppose that ∇f vanishes in the feasible set. True or False: We know we have an interior solution to the maximization of $f(x)$ subject to $g_j(x) \leq b_j$ for $j = 1, 2, 3$ (which lives in $\mathbb{R}^2_{++}$).

False. We may be at an inflection point; quasiconcavity is not strong enough to guarantee a maximum when ∇f vanishes.

7. (Total: 15 points) Consider the following NPP:

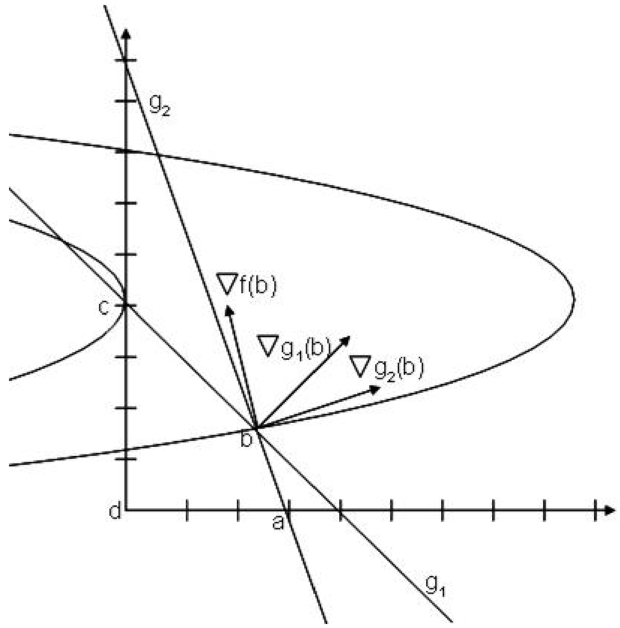
$$\max_{x_1,x_2} f(x_1, x_2) = -(x_1 - 2) - (x_2 - 4)^2 \tag{1}$$

$$g_1: \; x_2 \leq 4 - x_1 \qquad \text{and} \qquad g_2: \; x_2 \leq 9 - 3x_1$$

$$g_3: \; x_1 \geq 0 \qquad \text{and} \qquad g_4: \; x_2 \geq 0$$

Using graphical methods answer the following (Don't draw too small!):

(a) (3 points) Draw the (x_1, x_2) plane with the above constraints, indicate the feasible set.



The feasible set is the area formed by *abcd*.

(b) (3 points) In the same plot draw ∇g_1 and ∇g_2 where the constraint functions associated with g_1 and g_2 intersect (indicate this intersection with a 'b').

See the figure in 7(a).

(c) (2 points) In the same plot indicate with a 'c' the solution to the UNconstrained maximization of $f(x_1, x_2)$ above. What is ∇f evaluated at this point 'c' ?

There is no solution c to the unconstrained maximization problem and so $\nabla f(0, 4)$ is not defined. *Some of you treated the unconstrained maximization problem as only relaxing g_1 and g_2 and indicated a corner solution c at $(0, 4)$, and computed $\nabla f(0, 4)$. We gave full credit for this answer. The original intention was to have you consider the following objective function (the difference being the missing exponent on (x_1-2)): $\max_{x_1,x_2} f(x_1, x_2) = -(x_1-2)^2-(x_2-4)^2$. The unconstrained solution to this maximization problem is $(2, 4)$ and $\nabla f(2, 4) = [0, 0]'$.*

(d) (3 points) Draw a level set of $f(x_1, x_2)$ that passes through point 'b' and draw ∇f at point 'b'.

See the figure in 7(a). An easy check to see if your level set made sense was to consider $\nabla f(b)$, we know this should be perpendicular to the level set passing through b.

(e) (4 points) Using $\nabla g_1(b)$ and $\nabla g_2(b)$ as guides what does the positioning of $\nabla f(b)$ tell us about whether or not b is a solution to our constrained maximization problem? If b is not a solution then draw another level set of f to identify the solution to our constrained maximization problem.

$\nabla f(b)$ does not lie in the non-negative cone formed by $\nabla g_1(b)$ and $\nabla g_2(b)$. Based on $\nabla f(b)$ as drawn in the figure we know that the solution to our constrained maximization problem will lie on g_1; indeed, the solution is at $(0, 4)$–the corresponding level set has been drawn through this point.

8. (Total: 8 points) Consider the objective function: $f(x, \alpha) = \ln(x) - \alpha x^2$ with $\alpha > 0$.

   (a) (2 points) Derive the function $x^*(\cdot)$ which maps each possible α to the value of x that maximizes $f(\cdot, \alpha)$.

   From the first order conditions we find: $\frac{\partial f}{\partial x} = \frac{1}{x} - 2\alpha x = 0$. Re-arranging we find $x^*(\alpha) = \sqrt{\frac{1}{2\alpha}} = (2\alpha)^{-\frac{1}{2}}$.

   (b) (2 points) Given the above objective function, how will the optimum value of x change with respect to a change in α?

   We simply need to differentiate x^* w.r.t. α: $\frac{dx(\alpha)}{d\alpha} = -\frac{1}{2}(2\alpha)^{-\frac{3}{2}} \cdot 2 = -(2\alpha)^{-\frac{3}{2}}$.

   (c) (1 point) Replacing x with $x^*(\alpha)$ into our objective function above we obtain what is commonly called the value function.

   (d) (1 point) The total derivative of $f(x(\alpha), \alpha)$ w.r.t. to α is given by: $\frac{df}{d\alpha} = \frac{\partial f}{\partial \alpha} + \frac{\partial f}{\partial x(\alpha)} \cdot \frac{dx(\alpha)}{d\alpha}$. Compute the relevant partial derivatives and derivatives and replace them into the right-hand side of this equation (do not simplify).

   $\frac{df}{d\alpha} = -x(\alpha)^2 + (\frac{1}{x(\alpha)} - 2\alpha x(\alpha))(-(2\alpha)^{-(\frac{3}{2})})$.

   (e) (2 points) Using the envelope theorem simplify your answer in part (d).

   We know from the first order conditions that $\frac{\partial f}{\partial x} = \frac{1}{x} - 2\alpha x = 0$ so that the second term in 8(d) goes to zero, this leaves: $\frac{df}{d\alpha} = -x(\alpha)^2 = -\frac{1}{2\alpha}$.

9. (Total: 12 points) Consider a monopoly who has a constant marginal cost of production m and cannot produce more than Q units due to a capacity constraint (assume this constraint is binding). Also, let there be a per unit tax of τ. How does the equilibrium price and shadow value of capacity change with an increase in τ? (*If you have time, please complete the matrix algebra. Answers that leave the answer in matrix form will earn less than full credit*).

The monopolist's optimization problem:

$$\max_{p} \ (p-\tau)D(p) - mD(p) \qquad \text{s.t.} \qquad D(p) \leq Q$$

$$\mathcal{L}(p,\lambda;\tau) = (p-\tau)D(p) - mD(p) + \lambda(Q - D(p))$$

$$\mathcal{L}_p = D(p) + (p - \tau - m - \lambda)D'(p) = 0$$

$$\mathcal{L}_\lambda = Q - D(p) = 0$$

The Hessian of the Lagrangian is

$$\text{HL} = \begin{bmatrix} 2D'(p) + (p-\tau-m-\lambda)D''(p) & -D'(p) \\ -D'(p) & 0 \end{bmatrix}$$

While the derivative the of the first order conditions with respect to τ is

$$\begin{bmatrix} -D'(p) \\ 0 \end{bmatrix}$$

Hence, from the implicit function theorem, we have

$$\begin{bmatrix} \frac{dp^*}{d\tau} \\ \frac{d\lambda}{d\tau} \end{bmatrix} = -\text{HL}^{-1} \begin{bmatrix} -D'(p) \\ 0 \end{bmatrix}$$

Note that,

$$\det(\text{HL}) = -(D'(p))^2 < 0$$

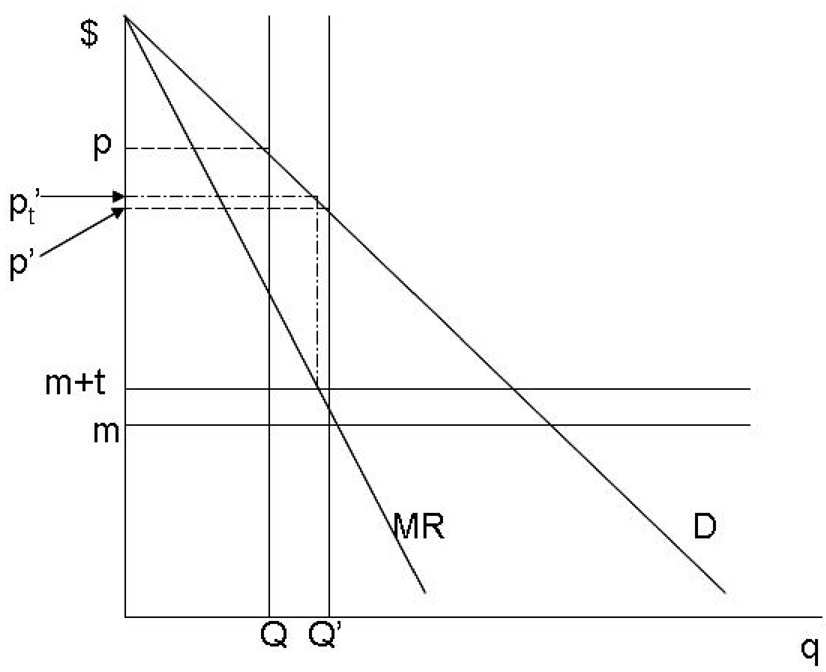
Applying Cramer's rule we have,

$$\frac{dp^*}{d\tau} = 0$$

$$\frac{d\lambda}{d\tau} = \det \begin{bmatrix} 2D'(p) + (p - \tau - m - \lambda)D''(p) & D'(p) \\ -D'(p) & 0 \end{bmatrix} / \det(\mathrm{HL}) = -1 < 0$$

The tax doesn't affect price but negatively affects the shadow value of capacity.

Extra musing on the last problem:
Graphical interpretation. When we do calculus we are thinking about an infinitesimally small change in the tax; hence, the capacity constraint will still end up binding after the tax. For example, in the figure below if Q is our capacity constraint, then even after the tax the price remains at p; price is unaffected. However, we see that the shadow value of capacity has been reduced by the size of the change in the per unit tax–that's why $\frac{d\lambda}{d\tau} = -1$.



Here is the problem with the calculus approach: Governments don't usually levy infinitesimally small taxes. For example, the tax may be large enough that the capacity constraint no longer binds. This is the case if the capacity constraint is Q' and the tax is t; the price moves from p' to p'_t.

Figure 1: $f:\ (0,7) \rightarrow S \in \mathbb{R}$

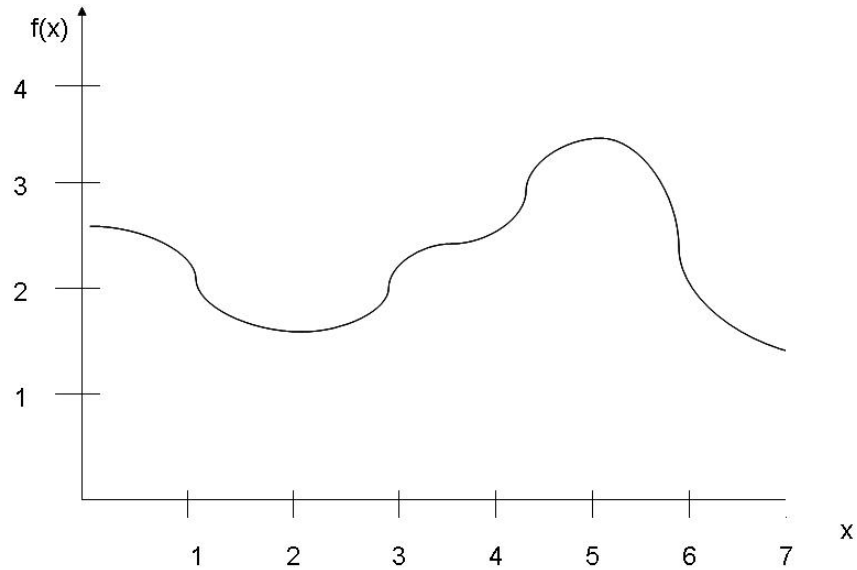


Figure 2: Domain of a continuously differentiable function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$