Final Exam

Tuesday, Dec 16, 2008

This is the final exam for ARE211. As announced earlier, this is an open-book exam. However, use of computers, calculators, Palm Pilots, cell phones, Blackberries and other non-human aids is forbidden.

Read all questions carefully before starting the test.

Allocate your 180 minutes in this exam wisely. The exam has 180 points, so aim for an *average* of 1 minute per point. However, some questions & parts are distinctly easier than others. Make sure that you first do all the easy parts, before you move onto the hard parts. Always bear in mind that if you leave a part-question completely blank, you cannot conceivably get any marks for that part. On the other hand, to give a 100% complete answer to all of the questions would almost certainly take you longer than the time available. (Writing the answer key took me quite a while.) In some problems, it may be efficient to simply write down the correct answer (if you know what it is) then come back and justify the answer later.

The questions are designed so that, to some extent, even if you cannot answer some parts, you will still be able to answer later parts. Even if you are unable to show a result, you are allowed to use the result in subsequent parts of the question. So don't hesitate to leave a part out. *if you get stuck on something, move on and come back later.* You don't have to answer questions and parts of questions in the order that they appear on the exam, *provided that you clearly indicate the question/part-question you are answering.* Finally, *always* keep in mind the famous maxim KISS (keep it simple, stupid).

Problem 1 (Real Analysis) [36 points]:

A) [*6 points*] Show that if $\{x_n\}$ is a convergent sequence, then for any N, the sequence given by the average $y_n = \frac{x_{n+1}+x_{n+2}+\ldots+x_{n+N}}{N}$ converges to the same limit. Is it possible for the sequence $\{y_n\}$ to converge even if $\{x_n\}$ does not?

B) [*6 points*] Show that if K is a compact subset of $\mathbb{R}$ and F is a closed subset of $\mathbb{R}$, then $K \cap F$ is compact.

C) [*7 points*] Prove that every finite set in $\mathbb{R}$ is compact.

D) [*7 points*] Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences in $\mathbb{R}$. Show that if $x_n \leq y_n \leq z_n$ for all n, and if $lim\{x_n\} = lim\{z_n\} = \ell$ then $lim\{y_n\} = \ell$ as well.

E) [*10 points*] Show that the empty set and $\mathbb{R}^n$ are both open and closed in $\mathbb{R}^n$. Prove that no other subsets of the $\mathbb{R}^n$ can be both open and closed.

Problem 2 (Linear Algebra) [36 points]:

A) [*3 points*] Give necessary and sufficient conditions for an $n \times m$ matrix A to be invertible.

B) [*3 points*] Verify that $\mathbf{v}^1 = \begin{bmatrix} 0.5 \\ \sqrt{3/4} \end{bmatrix}$ is a unit eigenvector of the matrix $A = \begin{bmatrix} 5 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{bmatrix}$.

C) [*3 points*] What is the eigenvalue corresponding to $\mathbf{v}^1$. Verify.

D) [*3 points*] Identify a second, distinct unit eigenvector $\mathbf{v}^2$ of the matrix A.

E) [*3 points*] What is the eigenvalue corresponding to $\mathbf{v}^2$. Verify.

F) [*3 points*] What can you say about the definiteness or otherwise of A?

G) [*5 points*] Prove that a 2×2 matrix A that has less than full rank is semi-definite.

H) [*3 points*] Now construct a matrix B with the following properties:
- all of its elements are non-zero
- it is symmetric
- all but one of its elements are equal to the corresponding elements of A
- it is semi-definite

I) [*5 points*] Is the matrix you have constructed positive or negative semi-definite?

J) [*5 points*] For $n > 2$, an $n \times n$ matrix that has less than full rank need not be semi-definite. Explain why not.

Problem 3 (Calculus) [36 points]:

You operate a factory that makes cars according to a production function, $F(K, L) = 3K^{1/3}L^{2/3}$.

A) [*5 points*] How many cars do you produce when your input mix is $(27, 125)$?

For the next four parts, use the answer you obtained in part A).

B) [*5 points*] Use the differential to compute a first order approximation to the number of cars you would produce if your input mix input were $(36, 130)$?

C) [*5 points*] What is the directional derivative of F $(27, 125)$ in the direction $(9, 5)$?

D) [*5 points*] Using the answer you obtained in part C), compute a first order approximation to the number of cars you would produce if your input mix input were $(36, 130)$?

E) [*6 points*] Compute a *second* order approximation to the number of cars you would produce if your input mix were $(36, 130)$? (Since you don't have calculators, we'll give you full credit for this part even if you don't complete the numerical computation. But you should go up to the point at which a calculator would be necessary.)

F) [*5 points*] In what proportions should you add K and L to $(27, 125)$ if you want to increase production most rapidly?

Your car company is unprofitable, so, in order to receive a government bailout, you must implement a new technology. Under this technology, your production inputs, K and L, are functions of time, t, and the interest rate, r. Specifically, $K(t, r) = 9\frac{t^2}{r}$ and $L(t, r) = t^2 + r$ (note: your production function, F, does not change).

G) [*5 points*] Calculate the rate of change of output with respect to t when $t = 10$ and $r = 0.1$.

Problem 4 (Kuhn Tucker) [36 points]:

Consider the problem

$$min_{x,y} \quad -3(x-10)(y-25)+(x-10)^3 \quad \text{s.t.} \quad 2x-y=-5, \quad 5x+2y\geq 37, \quad x\geq 0, \quad y\geq 0.$$

A) [*3 points*] Write this problem in the canonical NPP format.

B) [*3 points*] Sketch the constraint set.

C) [*3 points*] State what it means for the constraint qualification to be satisfied *in language that involves no mathematical symbols.*

D) [*3 points*] What points in the constraint set satisfy the constraint qualification?

At this point, to facilitate solving the problem, convert it into a problem with *one* unknown, x, and *one* constraint. Call the objective function f.

E) [*4 points*] Write down the single variable constrained optimization problem.

F) [*4 points*] Write down the first and second derivatives of f.

G) [*4 points*] Using your answer to part F), sketch the constraint set and objective function.

H) [*4 points*] What values of x satisfy the KKT conditions for this problem? (Get help from your graphical answer in part G).)

I) [*4 points*] What point on the constraint set solves the NPP? (Get help from your graphical answer in part G).)

J) [*4 points*] Comment on why there are more than one values of x that solve the KKT.

Problem 5 (Comparative Statics) [36 points]:

A) Consider the function $f(x,y,\gamma)=xy+\gamma y$ subject to the following constraints: $g(x,y,\gamma)\leq 1$, $x\geq 0$, $y\geq 0$, where $g(x,y)=x^2+\gamma y$.
   (a) [*10 points*] For $\gamma=1$, solve this maxmization problem using either the Lagrangian or KKT method.

   (b) [*10 points*] Now, use the envelope theorem to estimate the maximized value of f when $\gamma=1.2$

B) [*16 points*] Consider the problem $\max_x \; f(x;\alpha)$ s.t. $g(x;\alpha)\leq b$, where $x,\alpha,b\in\mathbb{R}$, f and g are twice continuously differentiable, $g_x(\cdot,\alpha)>0$. Let $x^*(\alpha)$ denote the solution to this problem, given α. Use the implicit function theorem to identify sufficient conditions for $x^*(\cdot)$ to be everywhere strictly increasing in α. Are the conditions you identified necessary as well? If so prove it. If not, provide a counter-example.