Final Exam

Thurs, Dec 14, 2006

This is the final exam for ARE211. As announced earlier, this is an open-book exam. However, use of computers, calculators, Palm Pilots, cell phones, Blackberries and other non-human aids is forbidden.

Read all questions carefully before starting the test.

Allocate your 180 minutes in this exam wisely. The exam has 180 points, so aim for 1 minute per point. Make sure that you first do all the easy parts, before you move onto the hard parts. Always bear in mind that if you leave a part-question completely blank, you cannot conceivably get any marks for that part. The questions are designed so that, to some extent, even if you cannot answer some parts, you will still be able to answer later parts. Even if you are unable to show a result, you are allowed to use the result in subsequent parts of the question. So don't hesitate to leave a part out. You don't have to answer questions and parts of questions in the order that they appear on the exam, *provided that you clearly indicate the question/part-question you are answering.*

Problem 1 [32 points]

A) Let $f(x) = (x-2)(x-1)(x+1)(x+2)$.
This function can be rewritten as:
$f(x) = (x^2-1)(x^2-4)$ or
$f(x) = x^4 - 5x^2 + 4$
Consider the NPP

$$\min_{x \in \mathbb{R}^1} \quad f(x) \quad \text{s.t.} \quad \begin{cases} x \geq -2 \\ x \leq 0.5 \end{cases}$$

(a) [*2 points*] Convert the problem to the standard format for an NPP that we have been using in this course.
(b) [*5 points*] Is the constraint qualification (CQ) satisfied at all the points in the constraint set?
(c) [*5 points*] Find the set of all points that satisfy the KKT conditions.
(d) [*5 points*] At what point is the minimum attained?

B) Now consider the problem

$$\max_{\mathbf{x} \in \mathbb{R}^2} \quad h(\mathbf{x}) \quad \text{s.t} \quad \begin{cases} x_2 \leq & (x_1-2)(x_1-1)(x_1+1)(x_1+2) \\ x_1 \geq & -2 \\ x_1 \leq & 1 \end{cases}$$

where $h(\mathbf{x}) = x_1 + x_2$. Do not solve this optimization problem!
(a) [*7 points*] Carefully apply KKT (including checking the CQ) for $\bar{\mathbf{x}} = (0, 4)$ and $\tilde{\mathbf{x}} = (1, 0)$.
(b) [*8 points*] Using *only* KKT, what can we conclude about $\bar{\mathbf{x}} = (0, 4)$ and $\tilde{\mathbf{x}} = (1, 0)$ as potential solutions to the optimization problem?

Problem 2 [32 points]
Consider the system of equations: $F(\mathbf{x}, \alpha) = S\mathbf{x} + G(\alpha) = \mathbf{0}$ where S is an invertible 2x2 matrix, G is a continuously differentiable function, and $\alpha \in \mathbb{R}^1$.

A) [*5 points*] Write down the domain and range of F.

B) [*9 points*] Treating α as a parameter, write down the Jacobian of F.

C) [*9 points*] Given any α, is the solution to the system of equations unique? If so, why? If not, give a counterexample.

D) [*9 points*] Given $G(\alpha) = \begin{bmatrix} \alpha^2 \\ \alpha^3 + (\alpha - 2)^2 \end{bmatrix}$ and $S = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$, let $\mathbf{x}^\star(\alpha)$ denote the solution to the equation system. Set $\alpha = 0$. Can you compute $\left(\frac{\partial x_1^\star(0)}{\partial \alpha}, \frac{\partial x_2^\star(0)}{\partial \alpha}\right)$ using the implicit function theorem? If so, do so; If not, explain which condition(s) of the theorem is violated.

Problem 3 [20 points]
Consider the function $F(\mathbf{x}) = A\mathbf{x} = \mathbf{0}$, where A is a $b \times c$ matrix with full rank.

A) If $b = c$, can we apply the implicit function theorem? If so, what other conditions are needed, if any?
B) If $b > c$, can we apply the implicit function theorem? If so, what other conditions are needed, if any?
C) If $b < c$, can we apply the implicit function theorem? If so, what other conditions are needed, if any?

Clarification 1: the elements of A are *not* variables. The column vector, $\mathbf{x}$, contains all the variables in the system.
Clarification 2: For the purposes of this question, define "full rank" as $\text{rank}(A) = \min(b, c)$.

Problem 4 [48 points]
Consider the following NPP:

$$\max_{\mathbf{x} \in \mathbb{R}^n} u(\mathbf{x}) \quad \text{s.t} \quad \begin{cases} x_i \geq b_i \text{ for } i = 1, \dots, n \\ \mathbf{p} \cdot \mathbf{x} \leq W \end{cases}$$

A) [*7 points*] Write down the Lagrangian for this problem. Then write down the $n + (n+1) + (n+1) + (n+1)$ KKT first order conditions in terms of the partial derivatives of the Lagrangian.

B) Let $n = 3$, $\mathbf{b} = (3, 6, 10)'$, $W = 60$, $\mathbf{p} = (3, 2, 1)'$, and $u(\mathbf{x}) = x_1^{0.5} x_2^{0.5}$.

(a) [*6 points*] Compute the solution to this NPP problem. Your answer should include values for the maximized value, the maximizing point, and the four Lagrange multipliers.

(b) [*6 points*] Compute a first order approximation of the change in the maximized value of utility if $\mathbf{b}$ changes by $d\mathbf{b} = (6, 6, 6)$.

C) Now consider $n = 2$, $\mathbf{b} = (\beta, 0)'$, $\mathbf{p} = (1, 1)'$, and $u(\mathbf{x}) = x_1^{0.5} x_2^{0.5}$. Let $0 < \beta \leq W$ be exogenously specified parameters. (Hint #1: a necessary condition for a maximum is that $x_i > 0$, for $i = 1, 2$. To see this, note that for any $\mathbf{x}$ such that $x_i = 0$, for some i, $u(\mathbf{x}) = 0$. Hint #2: since the objective is strictly pseudoconcave on $\mathbb{R}_{++}$ and the constraints are quasiconvex, the KKT conditions are sufficient for a solution.)

(a) [*6 points*] Draw the constraint set for $W = 60, \beta = 0$. Where is the maximum attained? You should be able to eye-ball the answer to this question. Once you have figured out the solution $\mathbf{x}$, plug this vector into your answer to A), and compute the values of all three multipliers.

(b) [*2 points*] Draw the constraint set for $W = 60, \beta = 50$. Where is the maximum attained? Again, you should be able to eye-ball the answer to this question. Again, once you have figured out the solution $\mathbf{x}$, plug this vector into your answer to A), and compute the values of all three multipliers. Your answer should include square root terms. Don't compute them, just write your answer as an expression in square roots. Simpler is better, but don't spend too much time simplifying.

(c) [*2 points*] For each W, there exists a unique scalar β^* with the following property: the solution to the NPP when $\beta' \in (\beta^*, W)$ is different from the solution to the NPP when $\beta' \in [0, \beta^*]$ (For example, with $W = 60$, given the solution you obtained to part C)(b) of this problem, you know that when $W = 60$, β^* must be between 0 and 50.) Calculate β^* for $W = 60$. To answer this part, you can either use a diagrammatic argument, or invoke the fact that the KKT conditions are sufficient for a solution.

(d) [*2 points*] For an arbitrary W, calculate β^*. To answer this part, you can either use a diagrammatic argument, or invoke the fact that the KKT conditions are sufficient for a solution.

(e) [*9 points*] Define $V(\beta, W)$ as the value function, i.e. the maximized value of the utility function, given the parameters β and W. Apply the envelope theorem to compute $\frac{\partial V}{\partial \beta}(\beta, W)$ and $\frac{\partial V}{\partial W}(\beta, W)$ for an arbitrary $W \in \mathbb{R}^1_{++}$ and $\beta \in [0, W]$. (Hint #1: The answer will require considering different cases. Hint #2: Check carefully that you have the signs right.)).

(f) [*8 points*] Compute the first order approximation to the change in V when the parameter vector increases from $(\beta, W) = (50, 60)$ to $(\beta, W)' = (60, 80)$. Again, don't compute out the values of square roots.

Problem 5 [48 points]
Consider the following economic system.

$$f : \mathbb{R} \to \mathbb{R} \text{ is defined by } f(x) = x, \text{ for all } x \in \mathbb{R}_+$$
$$g : \mathbb{R} \to \mathbb{R} \text{ satisfies the following properties}$$
$$g(0) > 0$$
$$\exists \epsilon > 0 \text{ such that } g''(\cdot) < -\epsilon$$

Both f and g are three times differentiable. An equilibrium for this system is defined as a scalar $\bar{x} \geq 0$ such that $f(\bar{x}) = g(\bar{x})$

A) [*2 points*] For which of the following functions, h, is the following statement true: (x is an equilibrium $\Leftrightarrow h(x) = 0$)?
A: $h = g - f$
B: $h = g + f$
Use the h you select in this part to answer the remaining parts of this questions.

B) [*5 points*] For $x > 0$, express $h(x)$ *exactly* in terms of the zero'th[1], first and second derivatives of h w.r.t. x. Except for the remainder term, evaluate the derivatives at $x = 0$.

C) [*5 points*] Prove that $h(1) < h(0) + h'(0) - 0.5\epsilon$.

D) [*2 points*] Is the following statement True or False?
Given $x > 0$, $\forall \lambda \in [0, 1]$ $g'(\lambda x) \geq g'(x)$.
Full credit for a one letter answer!

E) [*10 points*] Using Taylor theory, determine if $g'(\bar{x})$ is greater than, equal to, or less than 1. [Recommended approach: Draw a picture and look at your picture. Then consider a zero'th order Taylor expansion of g about 0.]

F) [*12 points*] Use the function h and some part of Taylor theory to prove that an equilibrium $\bar{x} > 0$ exists. [Recommended approach: Do a first order Taylor expansion of h about zero. The last line of your proof should include something like the following, which invokes a theorem known as the *intermediate value theorem*, which we haven't taught you but you can simply assert:
since $h(0) > 0$ and $\exists x > 0$ s.t. $h(x) < 0$, and h is continuous, an equilibrium must exist somewhere between 0 and x.]

G) [*12 points*] Use the function h and some part of Taylor theory to prove that this equilibrium is unique. [Recommended approach: Let X be the set of equilibria for this problem. (You can *assume* that X is a finite set.) Now let $\bar{x}$ be the smallest element of X and consider a first order Taylor expansion of h about $\bar{x}$. Finally, use your answer to E to show that $\bar{x}$ is the *only* element of X.]

[1] The zero'th derivative of f w.r.t. x is f