Final Exam - Answer key

Problem 1 [32 points]

A) Let $f(x) = (x-2)(x-1)(x+1)(x+2)$.
This function can be rewritten as:
$f(x) = (x^2-1)(x^2-4)$ or
$f(x) = x^4 - 5x^2 + 4$
Consider the NPP

$$\min_{x \in \mathbb{R}^1} \quad f(x) \quad \text{s.t.} \quad \begin{cases} x \geq -2 \\ x \leq 0.5 \end{cases}$$

(a) [*2 points*] Convert the problem to the standard format for an NPP that we have been using in this course.
(b) [*5 points*] Is the constraint qualification (CQ) satisfied at all the points in the constraint set?
(c) [*5 points*] Find the set of all points that satisfy the KKT conditions.
(d) [*5 points*] At what point is the minimum attained?

All points satisfy the CQ. The KKT conditions satisfied at $\{-\sqrt{5/2}, 0, 0.5\}$. The min is attained at $x = -\sqrt{5/2}$

B) Now consider the problem

$$\max_{\mathbf{x} \in \mathbb{R}^2} \quad h(\mathbf{x}) \quad \text{s.t} \quad \begin{cases} x_2 \leq & (x_1-2)(x_1-1)(x_1+1)(x_1+2) \\ x_1 \geq & -2 \\ x_1 \leq & 1 \end{cases}$$

where $h(\mathbf{x}) = x_1 + x_2$. Do not solve this optimization problem!

(a) [*7 points*] Carefully apply KKT (including checking the CQ) for $\bar{\mathbf{x}} = (0,4)$ and $\tilde{\mathbf{x}} = (1,0)$.
(b) [*8 points*] Using *only* KKT, what can we conclude about $\bar{\mathbf{x}} = (0,4)$ and $\tilde{\mathbf{x}} = (1,0)$ as potential solutions to the optimization problem?

For $\bar{\mathbf{x}} = (0,4)$, only the first constraint is satisfied with equality. $\nabla g_1(\bar{\mathbf{x}}) = (0,1)$. The CQ holds. $\nabla h(\bar{\mathbf{x}}) = (1,1)$. KKT necessary conditions not satisfied.
For $\tilde{\mathbf{x}} = (1,0)$, the first and third constraints are satisfied with equality. $\nabla g_1(\tilde{\mathbf{x}}) = (0,1)$. $\nabla g_3(\tilde{\mathbf{x}}) = (6,1)$. The CQ holds. $\nabla h(\tilde{\mathbf{x}}) = (1,1)$. KKT necessary conditions are not satisfied.
Using only the KKT, we can rule out both $\tilde{x}$ and $\bar{x}$ as possible maximizers.

Problem 2 [32 points]

Consider the system of equations: $F(\mathbf{x}, \alpha) = S\mathbf{x} + G(\alpha) = \mathbf{0}$ where S is an invertible 2x2 matrix, G is a continuously differentiable function, and $\alpha \in \mathbb{R}^1$.

A) [*5 points*] Write down the domain and range of F.

$f : \mathbb{R}^3 \rightarrow \mathbb{R}^2$

B) [*9 points*] Treating α as a parameter, write down the Jacobian of F.

Taking partials of F w.r.t. $\mathbf{x}$, the Jacobian of F, treating α as a parameter, is simply S.

C) [*9 points*] Given any α, is the solution to the system of equations unique? If so, why? If not, give a counterexample.

Solution is unique: S has full rank, so there exists a unique vector $\mathbf{x}$ such that $\mathbf{x} = -S^{-1}G(\alpha)$

D) [*9 points*] Given $G(\alpha) = \begin{bmatrix} \alpha^2 \\ \alpha^3 + (\alpha - 2)^2 \end{bmatrix}$ and $S = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$, let $\mathbf{x}^\star(\alpha)$ denote the solution to the equation system. Set $\alpha = 0$. Can you compute $\left(\frac{\partial x_1^\star(0)}{\partial \alpha}, \frac{\partial x_2^\star(0)}{\partial \alpha}\right)$ using the implicit function theorem? If so, do so; If not, explain which condition(s) of the theorem is violated.

It is possible. S is non-singular and hence invertible. $S^{-1} = \begin{bmatrix} 1 & -0.5 \\ 0 & 0.25 \end{bmatrix}$, and $JG(0) = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$. Hence $\begin{bmatrix} \frac{\partial x_1^\star(0)}{\partial \alpha} \\ \frac{\partial x_2^\star(0)}{\partial \alpha} \end{bmatrix} = -S^{-1}JG(0) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Problem 3 [20 points]

Consider the function $F(\mathbf{x}) = A\mathbf{x} = \mathbf{0}$, where A is a $b \times c$ matrix with full rank.

A) If $b = c$, can we apply the implicit function theorem? If so, what other conditions are needed, if any?

B) If $b > c$, can we apply the implicit function theorem? If so, what other conditions are needed, if any?

C) If $b < c$, can we apply the implicit function theorem? If so, what other conditions are needed, if any?

Clarification 1: the elements of A are *not* variables. The column vector, $\mathbf{x}$, contains all the variables in the system.

Clarification 2: For the purposes of this question, define "full rank" as $\text{rank}(A) = \min(b, c)$.

We can only apply the implicit function theorem if the number of equations equals the number of endogenous variables. In addition to the endogenous variables, there must be at least one exogenous variable. That is, the number of rows of A must be fewer than the number of columns i.e., $b < c$. If $b < c$, after picking which variables are endogenous, we need that the matrix keeping only the columns of the endogenous variables, is full rank. This is not guaranteed by A being of full rank, since for $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 6 & 9 \end{bmatrix}$, picking $\{x_2, x_3\}$ as endogenous does not work. An acceptable argument would be that given A is full rank, it is always possible to choose endogenous variables that will work, since column rank $=$ row rank. In the above example, both $\{x_1, x_3\}$ and $\{x_1, x_2\}$ would work as endogenous variables. Not discussing this issues at all means some points taken off.

Problem 4 [48 points]

Consider the following NPP:

$$\max_{\mathbf{x}\in\mathbb{R}^n} u(\mathbf{x}) \quad \text{s.t} \quad \begin{cases} x_i \geq b_i \text{ for } i = 1, \dots, n \\ \mathbf{p} \cdot \mathbf{x} \leq W \end{cases}$$

A) [*7 points*] Write down the Lagrangian for this problem. Then write down the $n + (n + 1) + (n + 1) + (n + 1)$ KKT first order conditions in terms of the partial derivatives of the Lagrangian.

$\mathcal{L}(\mathbf{x}, \lambda, \mu_i) = u(\mathbf{x}) + \lambda(W - \mathbf{p} \cdot \mathbf{x}) + \sum_{i=1}^{n} \mu_i(x_i - b_i)$

$$\text{FOC:} \begin{cases} \text{for } i = 1, \dots, n \quad \frac{\partial \mathcal{L}}{\partial x_i} = u_i(\mathbf{x}) - \lambda p_i + \mu_i = 0.5u(\mathbf{x})/x_i - \lambda p_i + \mu_i = 0 \\ \text{for } i = 1, \dots, n \quad \frac{\partial \mathcal{L}}{\partial \mu_i} = x_i - b_i \geq 0 \\ \text{for } i = 1, \dots, n \quad \mu_i \geq 0 \\ \text{for } i = 1, \dots, n \quad \mu_i \times \frac{\partial \mathcal{L}}{\partial \mu_i} = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} = W - \mathbf{p} \cdot \mathbf{x} \geq 0 \\ \lambda \geq 0 \\ \lambda \times \frac{\partial \mathcal{L}}{\partial \lambda} = 0 \end{cases}$$

All of these partials are evaluated at the point of interest.

B) Let $n = 3$, $\mathbf{b} = (3, 6, 10)'$, $W = 60$, $\mathbf{p} = (3, 2, 1)'$, and $u(\mathbf{x}) = x_1^{0.5} x_2^{0.5}$.

(a) [*6 points*] Compute the solution to this NPP problem. Your answer should include values for the maximized value, the maximizing point, and the four Lagrange multipliers.

$\nabla u(\mathbf{x}) = \begin{bmatrix} 0.5/x_1 \\ 0.5/x_2 \\ 0 \end{bmatrix} \times x_1^{0.5} x_2^{0.5}$. Hence, $p_1 x_1 = p_2 x_2 = \frac{0.5 x_1^{0.5} x_2^{0.5}}{\lambda}$ and $\lambda p_3 = \mu_3$. Solving all this out, we get, $\mathbf{x} = [50/6, \quad 50/4, \quad 10]$, $\lambda = 0.5/\sqrt{6}$, $\boldsymbol{\mu} = [0, \quad 0, \quad 0.5/\sqrt{6}]$.

(b) [*6 points*] Compute a first order approximation of the change in the maximized value of utility if $\mathbf{b}$ changes by $d\mathbf{b} = (6, 6, 6)$.

Apply the envelope theorem. $\nabla_b u(x(b)) = D_b u'(x(b)) = \begin{bmatrix} 0, & 0, & \mu_3 \end{bmatrix}$. The first order approximation to the change in u is $\begin{bmatrix} 0, & 0, & \mu_3 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix} = 3/\sqrt{6}$

C) Now consider $n = 2$, $\mathbf{b} = (\beta, 0)'$, $\mathbf{p} = (1, 1)'$, and $u(\mathbf{x}) = x_1^{0.5} x_2^{0.5}$. Let $0 < \beta \leq W$ be exogenously specified parameters. (Hint #1: a necessary condition for a maximum is that $x_i > 0$, for $i = 1, 2$. To see this, note that for any $\mathbf{x}$ such that $x_i = 0$, for some i, $u(\mathbf{x}) = 0$. Hint #2: since the objective is strictly pseudoconcave on $\mathbb{R}_{++}$ and the constraints are quasiconvex, the KKT conditions are sufficient for a solution.)

(a) [*6 points*] Draw the constraint set for $W = 60, \beta = 0$. Where is the maximum attained? You should be able to eye-ball the answer to this question. Once you have figured out the solution $\mathbf{x}$, plug this vector into your answer to A), and compute the values of all three multipliers.

Eyeballing, the max is attained at $\mathbf{x} = (30, 30)$. Since $x_i > b_i$, $\mu_i = 0$, for $i = 1, 2$. We now have $\frac{\partial \mathcal{L}}{\partial x_i} = 0.5u(\mathbf{x})/x_i - \lambda p_i + \mu_i = 0.5 \times 30/30 - \lambda = 0$, so that $\lambda = 0.5$.

(b) [*2 points*] Draw the constraint set for $W = 60, \beta = 50$. Where is the maximum attained? Again, you should be able to eye-ball the answer to this question. Again, once you have figured out the solution $\mathbf{x}$, plug this vector into your answer to A), and compute the values of all three multipliers. Your answer should include square root terms. Don't compute them, just write your answer as an expression in square roots. Simpler is better, but don't spend too much time simplifying.

Eyeballing, the max is attained at $\mathbf{x} = (50, 10)$. Since $10 > 0$, $\mu_2 = 0$.

$$\frac{\partial \mathcal{L}}{\partial x_2} = 0.5u(\mathbf{x})/x_2 - \lambda p_2 = 0.5\frac{\sqrt{500}}{10} - \lambda = \sqrt{1.25} - \lambda = 0;$$

Hence $\lambda = \sqrt{1.25}$. Finally,

$$\frac{\partial \mathcal{L}}{\partial x_1} = 0.5u(\mathbf{x})/x_1 - \lambda p_1 + \mu_1 = 0.5\frac{\sqrt{500}}{50} - \sqrt{1.25} + \mu_1 = \sqrt{0.05} - \sqrt{1.25} + \mu_1 = 0.$$

Hence $\mu_1 = \sqrt{1.25} - \sqrt{0.05}$.

(c) [*2 points*] For each W, there exists a unique scalar β^* with the following property: the solution to the NPP when $\beta' \in (\beta^*, W)$ is different from the solution to the NPP when $\beta' \in [0, \beta^*]$ (For example, with $W = 60$, given the solution you obtained to part C)(b) of this problem, you know that when $W = 60$, β^* must be between 0 and 50.) Calculate β^* for $W = 60$. To answer this part, you can either use a diagrammatic argument, or invoke the fact that the KKT conditions are sufficient for a solution.

$\beta^* = 30$

(d) [*2 points*] For an arbitrary W, calculate β^*. To answer this part, you can either use a diagrammatic argument, or invoke the fact that the KKT conditions are sufficient for a solution.

$\beta^*(W) = W/2$

(e) [*9 points*] Define $V(\beta, W)$ as the value function, i.e. the maximized value of the utility function, given the parameters β and W. Apply the envelope theorem to compute $\frac{\partial V}{\partial \beta}(\beta, W)$ and $\frac{\partial V}{\partial W}(\beta, W)$ for an arbitrary $W \in \mathbb{R}^1_{++}$ and $\beta \in [0, W]$. (Hint #1: The answer will require considering different cases. Hint #2: Check carefully that you have the signs right.)).

The answer depends on β. Applying the envelope theorem we have

$$\frac{\partial V}{\partial \beta}(\beta, W) = \begin{cases} 0 & \text{if } \beta < \beta^*(W) \\ -\mu_1 = -0.5\sqrt{\beta(W-\beta)}\left(\frac{1}{W-\beta} - \frac{1}{\beta}\right) & \text{if } \beta \geq \beta^*(W) \end{cases}$$

while

$$\frac{\partial V}{\partial W}(\beta, W) = \begin{cases} \lambda = 1/2 & \text{if } \beta < \beta^*(W) \\ \lambda = 0.5\sqrt{\beta(W-\beta)}/(W-\beta) & \text{if } \beta \geq \beta^*(W) \end{cases}$$

(f) [*8 points*] Compute the first order approximation to the change in V when the parameter vector increases from $(\beta, W) = (50, 60)$ to $(\beta, W)' = (60, 80)$. Again, don't compute out the values of square roots.

Plugging in numbers to the expressions we obtained in part (e), and noting that in each case, $\beta^* > W/2$, we have

$$\begin{aligned} \frac{\partial V}{\partial \beta}(\beta, W) &= -0.5\sqrt{500}\left(\frac{1}{10} - \frac{1}{50}\right) &= -0.5\frac{4}{50} \times 10\sqrt{5} = -\frac{2}{5}\sqrt{5} \\ \frac{\partial V}{\partial W}(\beta, W) &= 0.5\sqrt{500}/10 &= 0.5\sqrt{5} \end{aligned}$$

We can now evaluate the differential: $d\beta = 10$; $dW = 20$. Hence the first order approximation is

$$\frac{2}{5}\sqrt{5} \times 10 - 0.5\sqrt{5} \times 20 = -4\sqrt{5} + 10\sqrt{5} = 6\sqrt{5}$$

Problem 5 [48 points]
Consider the following economic system.

$$\begin{aligned} &f : \mathbb{R} \to \mathbb{R} \text{ is defined by } f(x) = x, \text{ for all } x \in \mathbb{R}_+ \\ &g : \mathbb{R} \to \mathbb{R} \text{ satisfies the following properties} \\ &\qquad g(0) > 0 \\ &\qquad \exists \epsilon > 0 \text{ such that } g''(\cdot) < -\epsilon \end{aligned}$$

Both f and g are three times differentiable. An equilibrium for this system is defined as a scalar $\bar{x} \geq 0$ such that $f(\bar{x}) = g(\bar{x})$

A) [*2 points*] For which of the following functions, h, is the following statement true: (x is an equilibrium $\Leftrightarrow h(x) = 0$)?
A: $h = g - f$
B: $h = g + f$
Use the h you select in this part to answer the remaining parts of this questions.

A i.e., $h = g - f$.

B) [*5 points*] For $x > 0$, express $h(x)$ *exactly* in terms of the zero'th[1], first and second derivatives of h w.r.t. x. Except for the remainder term, evaluate the derivatives at $x = 0$.

For some $\lambda \in [0, 1]$, $h(x) = h(0) + h'(0)x + 0.5h''(\lambda x)x^2$.

C) [*5 points*] Prove that $h(1) < h(0) + h'(0) - 0.5\epsilon$.

[1] The zero'th derivative of f w.r.t. x is f

From the previous part, $h(1) = h(0) + h'(0)1 + 0.5h''(\lambda x)1$. But by assumption, $h''(\lambda x) < -\epsilon$. Hence, $h(1) < h(0) + h'(0) - 0.5\epsilon$.

D) [*2 points*] Is the following statement True or False?
Given $x > 0$, $\forall \lambda \in [0, 1]$ $g'(\lambda x) \geq g'(x)$.
Full credit for a one letter answer!

True

E) [*10 points*] Using Taylor theory, determine if $g'(\bar{x})$ is greater than, equal to, or less than 1. [Recommended approach: Draw a picture and look at your picture. Then consider a zero'th order Taylor expansion of g about 0.]

There exists $\lambda \in [0, 1]$ such that

$$\bar{x} \quad = \quad f(\bar{x}) \quad = \quad g(\bar{x}) \quad = \quad g(0) + g'(\lambda\bar{x})\bar{x}$$

which, from the answer to part E) is

$$\geq \quad g(0) + g'(\bar{x})\bar{x}$$

Therefore, for some $\lambda \in [0, 1]$

$$g(0) + \big(g'(\lambda\bar{x}) - 1\big)\,\bar{x} = 0$$

Since $g(0) > 0$, it follows that $(g'(\lambda\bar{x}) - 1) < 0$, i.e., $g'(\lambda\bar{x}) < 1$.

F) [*12 points*] Use the function h and some part of Taylor theory to prove that an equilibrium $\bar{x} > 0$ exists. [Recommended approach: Do a first order Taylor expansion of h about zero. The last line of your proof should include something like the following, which invokes a theorem known as the *intermediate value theorem*, which we haven't taught you but you can simply assert:
since $h(0) > 0$ and $\exists x > 0$ s.t. $h(x) < 0$, and h is continuous, an equilibrium must exist somewhere between 0 and x.]

Given $dx > 0$, there exists $\lambda \in [0, 1]$ such that $h(dx) = h(0) + h'(0)dx + 0.5h''(\lambda dx)dx^2$. Let $dx > \max[h(0)/h'(0), 4h'(0)/\epsilon]$ and note that since $h''(\lambda dx) = g''(\lambda dx) - f''(\lambda dx) = g''(\lambda dx) < -\epsilon$,

$$\begin{aligned} h(dx) &= h(0) + h'(0)dx + 0.5h''(\lambda dx)dx^2 \\ &< h(0) + (h'(0) - 0.5\epsilon(4h'(0)/\epsilon))dx \\ &= h(0) + (h'(0) - 2h'(0))dx \\ &= h(0) - h'(0)dx \\ &< h(0) - h(0) = 0 \end{aligned}$$

Thus, for $x = dx$, $h(x) < 0$. Since $h(0) > 0$ and $\exists x > 0$ s.t. $h(x) < 0$, and h is continuous, an equilibrium must exist somewhere between 0 and x.

G) [*12 points*] Use the function h and some part of Taylor theory to prove that this equilibrium is unique. [Recommended approach: Let X be the set of equilibria for this problem. (You can *assume* that X is a finite set.) Now let $\bar{x}$ be the smallest element of X and consider a first order Taylor expansion of h about $\bar{x}$. Finally, use your answer to E to show that $\bar{x}$ is the *only* element of X.]

For arbitrarily chosen $x = \bar{x} + dx$, $dx > 0$, there exists $\lambda > 0$ s.t.

$$h(\bar{x} + dx) = h(\bar{x}) + h'(\bar{x})dx + 0.5h''(\bar{x} + \lambda dx)dx^2$$

Now, $h(\bar{x}) = 0$, and $h'(\cdot) = g'(\cdot) - 1 < 0$ (from E) while $h''(\cdot) < -\epsilon$, so that

$$h(\bar{x} + dx) < 0 + (g'(\bar{x}) - 1)dx - \epsilon dx^2.$$

Since dx is positive by assumption, $h(\bar{x} + dx) < 0$ and so cannot belong to X.