# Final Exam

Thurs, Dec 15, 2005

This is the final exam for ARE211. As announced earlier, this is an open-book exam. However, use of computers, calculators, Palm Pilots, cell phones, Blackberries and other non-human aids is forbidden.

Read all questions carefully before starting the test.

Allocate your 180 minutes in this exam wisely. The exam has 180 points, so aim for 1 minute per point. Make sure that you first do all the easy parts, before you move onto the hard parts. Always bear in mind that if you leave a part-question completely blank, you cannot conceivably get any marks for that part. The questions are designed so that, to some extent, even if you cannot answer some parts, you will still be able to answer later parts. Even if you are unable to show a result, you are allowed to use the result in subsequent parts of the question. So don't hesitate to leave a part out. You don't have to answer questions and parts of questions in the order that they appear on the exam, *provided that you clearly indicate the question/part-question you are answering.*

**Choose *either* Problem #1 *or* Problem #2.**

**Problem 1 [30 points]**
Let $E$ be a convex subset of $\mathbb{R}^n$, for some $n > 0$. Let $f : E \to \mathbb{R}$. Consider the following two definitions.
*Definition 1:* $f$ is ... if

$$\forall\,(x,y) \in E^2, x \neq y, \forall\,\theta \in (0,1), f(\theta x + (1-\theta)y) > \theta f(x) + (1-\theta)f(y).$$

*Definition 2:* $f$ is ... if

$$\forall\,(x,y) \in E^2, x \neq y, \forall\,\theta \in (0,1), f(x) \geq f(y) \Rightarrow f(\theta x + (1-\theta)y) > f(y).$$

A) [*5 points*] What does it mean for $E$ to be a convex subset of $\mathbb{R}^n$?

B) [*5 points*] Fill in the blanks in the statements of the definitions.

C) [*20 points*] Prove that if $f$ satisfies Definition 1, then $f$ also satisfies Definition 2.

**Problem 2 [30 points]**
Define $f : \mathbb{R}^2 \to \mathbb{R}$ such that

$$f(x,y) = \begin{cases} \frac{x^3 \ln|x|}{ye^y} & if\ x \neq 0,\ y \neq 0 \\ 0 & otherwise \end{cases}.$$

A) [*12 points*] Compute $\nabla f(0,0)$, if it exists. (Hint: the derivative of $e^x$ on $\mathbb{R}$ is $e^x$; the derivative of $\ln x$ w.r.t. $x$ is $1/x$.)

B) [*12 points*] Compute the directional derivative of $f$ at $(0,0)$ in direction $(1,1)$, if it exists.

C) [*6 points*] From your answers to the previous parts, what can you conclude about the differentiability of $f$ at $(0,0)$?

**Problem 3 [90 points]**
(Points do not include the optional bonus parts)

Fix $\alpha \geq 1$ and consider the following class of maximization problems, denoted NPP[$\alpha$]:

$$\max_{x,y} xe^y \quad sub.\,to \quad \begin{cases} \alpha x^2 + 2xy + y^2 \leq 1 \\ x \geq 0 \\ y \geq 0 \end{cases},$$

where $e^x$ denotes the exponential function (Hint: the derivative of $e^x$ on $\mathbb{R}$ is $e^x$).

A) [*5 points*] Briefly explain why problem NPP[$\alpha$] has at least one solution.

B) **Optional Bonus Part** [*10 points*] Show that the objective function is strictly quasi-concave.

Hint: You may use the following theorem:

Let $M$ be an $N \times N$ symmetric matrix and let $B$ be an $S \times N$ matrix with $S \leq N$ and rank equal to $S$. $M$ is negative definite on $\{z \in \mathbb{R}^N | Bz = 0\}$ if and only if

$$(-1)^r \begin{vmatrix} {}_rM_r & B_r^T \\ B_r & 0 \end{vmatrix} > 0$$

for $r = S+1, ..., N$, where ${}_rM_r$ represents the matrix obtained by deleting the last $(n-r)$ rows and columns of $M$ and $B_r$ represents the matrix obtained by deleting the last $(n-r)$ columns of $B$.

C) [*5 points*] You can take for granted that the constraint set is convex. Using your answers to the previous parts, can you be sure that the solution to problem NPP[$\alpha$] is unique?

D) [*5 points*] Write down the problem NPP[1] and show that for this problem all constraints are linear. Explain why, at the solution, $x$ must be positive.

E) [*5 points*] Write down the Lagrangian for problem NPP[1]. You may omit from the Lagrangian expression constraints that you *know* can never be satisfied with equality at the solution.

F) [*10 points*] Derive the F.O.C. to problem NPP[1] and solve them. In particular, explain why the 2 relevant constraints must be satisfied with equality at the solution. Also check that the constraint qualification is satisfied at all potential solutions.

G) [*5 points*] Write down a Lagrangian for problem NPP[$\alpha$]. You may omit from the Lagrangian expression constraints that you *know* can never be satisfied with equality at the solution.

H) [*5 points*] Derive the F.O.C. to problem NPP[$\alpha$].

I) [*5 points*] Show that the gradient of the objective function never vanishes.

J) [*5 points*] Using your answer to parts B) and I), explain why the F.O.C. derived in part H) are sufficient for a global solution to problem NPP[$\alpha$].

K) [*10 points*] Show that, for $\alpha > 1$, $(x, y) = (1, 0)$ cannot be a solution to problem NPP[$\alpha$].

L) [*10 points*] Using your answer to part H), show that if $\alpha > 1$, we must have $y > 0$ at a solution.
Hint: Suppose that $y = 0$ and find a contradiction.

M) [*5 points*] Show that when $\alpha \geq 1$, the unique solution to problem NPP[$\alpha$] is characterized by the following set of equations:

$$\begin{cases} \alpha x^2 + 2xy + y^2 - 1 = 0 \\ 2\alpha x^2 + 2xy - 2x - 2y = 0 \end{cases} .$$

N) [*10 points*] Using the Implicit Function Theorem, compute $\begin{bmatrix} \frac{dx^*}{d\alpha} \\ \frac{dy^*}{d\alpha} \end{bmatrix}$ at $\alpha = 1$, where $(x^*(\alpha), y^*(\alpha))$ denotes the unique solution to problem NPP[$\alpha$].

O) [*5 points*] Using the Envelope Theorem, approximate the value of the value function for problem NPP[1.1].

P) **Optional Bonus Part** [*5 points*] Using the fact that the value function of problem NPP[$\alpha$] is equal to $x^*(\alpha)e^{y^*(\alpha)}$, answer part O) by using the Chain Rule instead of the Envelope Theorem. Check that you obtain the same answer as you did when you used the Envelope Theorem.

**Problem 4 [30 points]**
Let $r$ denote the Euclidean (a.k.a Pythagorian) metric on $\mathbb{R}^n \times \mathbb{R}^n$ and let $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be defined by, for some $\alpha \in \mathbb{R}_{++}$ and all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

$$d(\mathbf{x}, \mathbf{y}) \quad = \quad \begin{cases} \alpha + r(\mathbf{x}, \mathbf{y}) & \text{if } \mathbf{x} \neq \mathbf{y} \\ 0 & \text{otherwise} \end{cases}.$$

Let $\rho$ denote an arbitrary metric on $\mathbb{R}^m$. Now define the functions $\psi^+ : \mathbb{R}^{n+m} \times \mathbb{R}^{n+m} \to \mathbb{R}$ and $\psi^- : \mathbb{R}^{n+m} \times \mathbb{R}^{n+m} \to \mathbb{R}$ by, for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{n+m}$

$$\begin{aligned} \psi^+(\mathbf{x}, \mathbf{y}) \quad &= \quad d\big((x_1, ..., x_n), (y_1, ..., y_n)\big) \quad + \quad \rho\big((x_{n+1}, ..., x_{n+m}), (y_{n+1}, ..., y_{n+m})\big) \\ \psi^-(\mathbf{x}, \mathbf{y}) \quad &= \quad \big|d\big((x_1, ..., x_n), (y_1, ..., y_n)\big) \quad - \quad \rho\big((x_{n+1}, ..., x_{n+m}), (y_{n+1}, ..., y_{n+m})\big)\big| \end{aligned}$$

A) [*7 points*] Prove that $\psi^+$ is a metric on $\mathbb{R}^{n+m}$

B) [*7 points*] Is $\psi^-$ a metric on $\mathbb{R}^{n+m}$ for all possible specifications of $r$, $\rho$ and $\alpha$? If so, prove it, if not provide a counterexample. (Hint: in thinking about this problem, it is helpful to begin with the simplest of all metrics...)

C) [*7 points*] Write down a necessary condition for a sequence to be a convergent sequence in $\mathbb{R}^{n+m}$ with respect to $\psi^+$. Prove that it is necessary. (The following is *not* an acceptable answer: a necessary condition is that the first $n$ components of the vectors converge w.r.t. $d$ and the last $m$ converge w.r.t. $\rho$.)

D) [*9 points*] Write down a sufficient but not necessary condition for a sequence to be a convergent sequence in $\mathbb{R}^{n+m}$ with respect to $\psi^+$. Prove that it is sufficient, and show that it is not necessary. (The following is *not* an acceptable answer: a sufficient condition is that the first $n$ components of the vectors converge w.r.t. $d$ and the last $m$ converge w.r.t. $\rho$.)

**Problem 5 [30 points]**
Pick $a \in \mathbb{R}$ and let $(x_n)$ be a sequence in $X \subset [-a, a]$, satisfying the following property: for all $n \in \mathbb{N}$, there exists $m > n$ such that $x_m \geq x_n$.

A) [*7 points*] Write down a nondecreasing subsequence of $(x_n)$. You should define it inductively. (Hint: a useful fact is that every subset of $\mathbb{N}$ has a smallest element).

B) [*23 points*] Suppose that $(x_n)$ contains no convergent subsequence. Prove that the set $X$ is not closed in $\mathbb{R}$.