Fall2005 **ARE211**

FINAL EXAM - ANSWER KEY

Problem 1 [30 points]

Let E be a convex subset of $\mathbb{R}^n$, for some $n > 0$. Let $f : E \to \mathbb{R}$. Consider the following two definitions.

Definition 1: f is ... if

$$\forall (x,y) \in E^2, x \neq y, \forall \theta \in (0,1), f(\theta x + (1-\theta)y) > \theta f(x) + (1-\theta) f(y).$$

Definition 2: f is ... if

$$\forall (x,y) \in E^2, x \neq y, \forall \theta \in (0,1), f(x) \geq f(y) \Rightarrow f(\theta x + (1-\theta)y) > f(y).$$

A) [*5 points*] What does it mean for E to be a convex subset of $\mathbb{R}^n$?

If E is a subset of $\mathbb{R}^n$, it means that E is included in $\mathbb{R}^n$. If E is convex, it means that the line segment joining any two elements of E (i.e., the set of convex combinations of these two elements) is included in E.

B) [*5 points*] Fill in the blanks in the statements of the definitions.

strictly concave, strictly quasiconcave.

C) [*20 points*] Prove that if f satisfies Definition 1, then f also satisfies Definition 2.

Suppose that f satisfies Definition 1. If $E = \emptyset$, then Definition 2 is satisfied trivially. If E has only one element, then Definition 2 also satisfied trivially. Thus, suppose that E has at least two elements, and choose (x,y) in E^2 such that $x \neq y$. Choose $\theta \in (0,1)$, and suppose that $f(x) \geq f(y)$. Note that since E is convex, $\theta x + (1-\theta)y$ is in E and $f(\theta x + (1-\theta)y)$ is defined. Want to prove: $f(\theta x + (1-\theta)y) > f(y)$. Since f satisfies Definition 1, we know that for our choice of (x,y) and θ, $f(\theta x + (1-\theta)y) > \theta f(x) + (1-\theta) f(y)$. Since $f(x) \geq f(y)$, we must have $f(\theta x + (1-\theta)y) > \theta f(y) + (1-\theta) f(y) = f(y)$, which completes the proof.

Problem 2 [30 points]
Define $f : \mathbb{R}^2 \to \mathbb{R}$ such that

$$f(x,y) = \begin{cases} \frac{x^3 \ln|x|}{ye^y} & if\ x \neq 0,\ y \neq 0 \\ 0 & otherwise \end{cases}.$$

A) [*12 points*] Compute $\nabla f(0,0)$, if it exists. (Hint: the derivative of e^x on $\mathbb{R}$ is e^x; the derivative of $\ln x$ w.r.t. x is $1/x$.)

f is not a "usual" function on its domain since it has two different expressions according to whether x and y are equal to zero or not. Therefore, to compute $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$, we cannot use the usual chain rule theorem and need to go back to the definition of partial derivatives.
For $t \neq 0$, $\frac{f(t,0)-f(0,0)}{t} = \frac{0-0}{t} = 0$. Hence,

$$\lim_{t\to 0} \frac{f(t,0)-f(0,0)}{t} = 0$$

and $\frac{\partial f}{\partial x}(0,0)$ exists and is equal to zero.
Similarly, for $t \neq 0$, $\frac{f(0,t)-f(0,0)}{t} = \frac{0-0}{t} = 0$. Hence,

$$\lim_{t\to 0} \frac{f(0,t)-f(0,0)}{t} = 0$$

and $\frac{\partial f}{\partial y}(0,0)$ exists and is equal to zero. Therefore, we can write

$$\nabla f(0,0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

B) [*12 points*] Compute the directional derivative of f at $(0,0)$ in direction $(1,1)$, if it exists.

We need to study the limit of the ratio $\frac{f((0,0)+\frac{(1,1)}{k})-f(0,0)}{\frac{||(1,1)||}{k}}$ as $|k| \to \infty$.

$$\frac{f((0,0)+\frac{(1,1)}{k})-f(0,0)}{\frac{||(1,1)||}{k}} = \frac{f(\frac{1}{k},\frac{1}{k})}{\frac{\sqrt{2}}{k}} = \frac{k}{\sqrt{2}} \frac{(\frac{1}{k})^3 \ln|\frac{1}{k}|}{\frac{1}{k}e^{\frac{1}{k}}} = \frac{k}{\sqrt{2}} \frac{(\frac{1}{k})^2 \ln|\frac{1}{k}|}{e^{\frac{1}{k}}} = \frac{\frac{1}{k}\ln|\frac{1}{k}|}{\sqrt{2}e^{\frac{1}{k}}}.$$

Since $\lim_{t\to 0} t\ln t = 0$, this ratio goes to 0 when $|k| \to \infty$. Therefore, the directional derivative of f at $(0,0)$ in direction $(1,1)$ exists and is equal to zero.

C) [*6 points*] From your answers to the previous parts, what can you conclude about the differentiability of f at $(0,0)$?

Nothing, because we only looked at the directional derivative of f at $(0,0)$ in one direction.

Problem 3 [90 points]
(Points do not include the optional bonus parts)

Fix $\alpha \geq 1$ and consider the following class of maximization problems, denoted NPP[α]:

$$\max_{x,y} xe^y \quad sub. to \quad \begin{cases} \alpha x^2 + 2xy + y^2 \leq 1 \\ x \geq 0 \\ y \geq 0 \end{cases},$$

where e^x denotes the exponential function (Hint: the derivative of e^x on $\mathbb{R}$ is e^x).

A) [*5 points*] Briefly explain why problem NPP[α] has at least one solution.

The constraint set is compact because of the weak inequalities and the fact that the constraint $\alpha x^2 + 2xy + y^2 \leq 1$ implies that the constraint set is bounded. Since the objective function is clearly continuous, by the Extremum Value Theorem it has a global max and a global min on the constraint set.

B) **Optional Bonus Part** [*10 points*] Show that the objective function is strictly quasi-concave.

Hint: You may use the following theorem:

Let M be an $N \times N$ symmetric matrix and let B be an $S \times N$ matrix with $S \leq N$ and rank equal to S. M is negative definite on $\{z \in \mathbb{R}^N | Bz = 0\}$ if and only if

$$(-1)^r \begin{vmatrix} {}_rM_r & B_r^T \\ B_r & 0 \end{vmatrix} > 0$$

for $r = S+1, ..., N$, where ${}_rM_r$ represents the matrix obtained by deleting the last $(n-r)$ rows and columns of M and B_r represents the matrix obtained by deleting the last $(n-r)$ columns of B.

A sufficient (but not necessary) condition for f to be strictly quasi-concave on $\mathbb{R}_{++} \times \mathbb{R}_+$ is that for all $(x, y) \in \mathbb{R}_{++} \times \mathbb{R}_+$, the Hessian of f at (x, y), denoted $D^2 f(x, y)$, is negative definite on the subset $\{z \in \mathbb{R}^2 | \nabla f(x) \cdot z = 0\}$. We have:

$$\nabla f(x, y) = \begin{pmatrix} e^y \\ xe^y \end{pmatrix}$$

and

$$D^2 f(x, y) = \begin{pmatrix} 0 & e^y \\ e^y & xe^y \end{pmatrix}.$$

Hence, using the bordered Hessian characterization given in the hint, a sufficient condition for f to be strictly quasi-concave will be that

$$\begin{vmatrix} 0 & e^y & e^y \\ e^y & xe^y & xe^y \\ e^y & xe^y & 0 \end{vmatrix} > 0.$$

By expanding along the first column, we find that this determinant is equal to xe^{3y}, which is strictly positive for $(x, y) \in \mathbb{R}_{++} \times \mathbb{R}_{+}$.

C) [*5 points*] You can take for granted that the constraint set is convex. Using your answers to the previous parts, can you be sure that the solution to problem NPP$[\alpha]$ is unique?

Since the constraint set is convex and the objective function is strictly quasi-concave, we can be sure that the solution is unique.

D) [*5 points*] Write down the problem NPP[1] and show that for this problem all constraints are linear. Explain why, at the solution, x must be positive.

When $\alpha = 1$, the first constraint becomes $x^2+2xy+y^2 \leq 1$, which can be written as $(x+y)^2 \leq 1$. Since $x \geq 0$ and $y \geq 0$, this is also equivalent to $x + y \leq 1$. Hence, the problem $eMAX[1]$ can be written as

$$\max_{x,y} xe^y \quad sub.\,to \quad \begin{cases} x+y \leq 1 \\ x \geq 0 \\ y \geq 0 \end{cases}.$$

Besides, since the objective function clearly attains positive values on the constraint set while it is always zero when $x = 0$, at a solution it must be that $x > 0$

E) [*5 points*] Write down the Lagrangian for problem NPP[1]. You may omit from the Lagrangian expression constraints that you *know* can never be satisfied with equality at the solution.

$$\mathcal{L}_1(x, y, \lambda, \mu) = xe^y + \lambda(1 - x - y) + \mu y$$

F) [*10 points*] Derive the F.O.C. to problem NPP[1] and solve them. In particular, explain why the 2 relevant constraints must be satisfied with equality at the solution. Also check that the constraint qualification is satisfied at all potential solutions.

The F.O.C. are:

$$\begin{cases} e^y - \lambda = 0 \\ xe^y - \lambda + \mu = 0 \\ \lambda(x+y-1) = 0 \\ \mu y = 0 \\ x + y \leq 1 \\ y \geq 0 \\ \lambda \geq 0 \\ \mu \geq 0 \end{cases}.$$

Plugging $\lambda = e^y$ into the second equality, we obtain $e^y(x-1)+\mu = 0$. Since $e^y > 0$ for all y, if $x < 1$ then we must have $\mu > 0$, which in turns implies that $y = 0$. But then $x + y < 1$, so that $\lambda = 0$. This contradicts the fact that $\lambda = e^y$. Therefore, it must be that $x = 1$. The bundle $(x, y, \lambda, \mu) = (1, 0, 1, 0)$ satisfies the F.O.C. (and is the only one to do so).

Clearly, the gradients of the two relevant constraints are always non colinear, so that the C.Q. must hold at all potential solutions.

G) [*5 points*] Write down a Lagrangian for problem NPP[α]. You may omit from the Lagrangian expression constraints that you *know* can never be satisfied with equality at the solution.

$$\mathcal{L}_\alpha(x, y, \lambda, \mu) = xe^y + \lambda(1 - \alpha x^2 - 2xy - y^2) + \mu y,$$

since at a solution it must be that $x > 0$.

H) [*5 points*] Derive the F.O.C. to problem NPP[α].

The F.O.C. are:

$$\begin{cases} e^y - \lambda(2\alpha x + 2y) = 0 \\ xe^y - \lambda(2x + 2y) + \mu = 0 \\ \lambda(\alpha x^2 + 2xy + y^2 - 1) = 0 \\ \mu y = 0 \\ \alpha x^2 + 2xy + y^2 \leq 1 \\ y \geq 0 \\ \lambda \geq 0 \\ \mu \geq 0 \end{cases}.$$

I) [*5 points*] Show that the gradient of the objective function never vanishes.

The gradient of the objective function at (x, y) is

$$\nabla f(x, y) = \begin{bmatrix} e^y \\ xe^y \end{bmatrix},$$

which is never zero.

J) [*5 points*] Using your answer to parts B) and I), explain why the F.O.C. derived in part H) are sufficient for a global solution to problem NPP[α].

The constraint set is convex, the objective function is quasi-concave (in fact, even strictly quasi-concave) and the gradient never vanishes. Hence, the K-T conditions are sufficient for a global maximizer.

K) [*10 points*] Show that, for $\alpha > 1$, $(x, y) = (1, 0)$ cannot be a solution to problem NPP[α].

Suppose that $\alpha > 1$. Let us plug $(x, y) = (1, 0)$ into the F.O.C. and see if we can find (λ, μ) such that the F.O.C. are satisfied. The first equation gives $\lambda = \frac{1}{2\alpha}$, which when plugged into the second equation yields $\mu = -1 + \frac{1}{\alpha} = \frac{1-\alpha}{\alpha}$. Hence, if $\alpha > 1$, we cannot find any (λ, μ) such that the bundle $(1, 0, \lambda, \mu)$ satisfies the F.O.C.

L) [*10 points*] Using your answer to part H), show that if $\alpha > 1$, we must have $y > 0$ at a solution. Hint: Suppose that $y = 0$ and find a contradiction.

Fix $\alpha > 1$ and suppose that $y = 0$. Then, the first equation of the F.O.C. implies that $\lambda = \frac{1}{2\alpha x}$, which when plugged into the second equation yields $\mu = -x + \frac{1}{\alpha}$. Since the first equation implies that $\lambda > 0$, the third equation implies that $\alpha x^2 - 1 = 0$, which is equivalent to $x = \sqrt{\frac{1}{\alpha}}$. Since, for $\alpha > 1$, $\sqrt{\frac{1}{\alpha}} > \frac{1}{\alpha}$, we must have $\mu < 0$, which contradicts the last inequality of the F.O.C. Therefore, since the F.O.C. are necessary and sufficient, it must be that at the solution $y > 0$.

M) [*5 points*] Show that when $\alpha \geq 1$, the unique solution to problem NPP[α] is characterized by the following set of equations:

$$\left\{ \begin{array}{l} \alpha x^2 + 2xy + y^2 - 1 = 0 \\ 2\alpha x^2 + 2xy - 2x - 2y = 0 \end{array} \right. .$$

The first equality is a consequence of the fact that $\lambda > 0$. The second equality can be obtained by plugging $\lambda = \frac{e^y}{2\alpha x + 2y}$ into the second equality and multiplying it by $\frac{2\alpha x + 2y}{e^y} \neq 0$. Note that since we have showed that at a solution $y > 0$, we must have $\mu = 0$.

N) [*10 points*] Using the Implicit Function Theorem, compute $\begin{bmatrix} \frac{dx^*}{d\alpha} \\ \frac{dy^*}{d\alpha} \end{bmatrix}$ at $\alpha = 1$, where $(x^*(\alpha), y^*(\alpha))$ denotes the unique solution to problem NPP[α].

The Jacobian matrix of the system w.r.t. (x, y) is

$$J_{x,y} = \begin{bmatrix} 2\alpha x + 2y & 2x + 2y \\ 4\alpha x + 2y - 2 & 2x - 2 \end{bmatrix}.$$

Evaluated at $\alpha = 1$ and $(x^*(1), y^*(1)) = (1, 0)$, this Jacobian matrix becomes

$$J_{x,y} = \begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix},$$

which is clearly nonsingular. Hence, the Implicit Function Theorem gives

$$\begin{aligned} \begin{bmatrix} \frac{dx^*}{d\alpha} \\ \frac{dy^*}{d\alpha} \end{bmatrix} &= -J_{x,y}^{-1} J_\alpha \\ &= -\frac{1}{-4} \begin{bmatrix} 0 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ \frac{1}{2} \end{bmatrix}. \end{aligned} \tag{1}$$

O) [*5 points*] Using the Envelope Theorem, approximate the value of the value function for problem NPP[1.1].

Call $v(\alpha)$ the value function of problem $eMAX[\alpha]$. By the Envelope Theorem, we have

$$v'(1) = 0 - \lambda(x^*(1))^2 = -\frac{1}{2}.$$

Using a first-order Taylor approximation, we find that

$$v(1.1) \approx v(1) + v'(1)(.1) = 1 - \frac{1}{2}(.1) = .95.$$

P) **Optional Bonus Part** [*5 points*] Using the fact that the value function of problem NPP[α] is equal to $x^*(\alpha)e^{y^*(\alpha)}$, answer part O) by using the Chain Rule instead of the Envelope Theorem. Check that you obtain the same answer as you did when you used the Envelope Theorem.

Since $v(\alpha) = x^*(\alpha)e^{y^*(\alpha)}$, by the Chain Rule we have that

$$v'(\alpha) = e^{y^*} \frac{dx^*}{d\alpha} + x^* e^{y^*} \frac{dy^*}{d\alpha}.$$

Hence,

$$v'(1) = \frac{dx^*}{d\alpha}(1) + \frac{dy^*}{d\alpha}(1) = -1 + \frac{1}{2} = -\frac{1}{2}.$$

Since we find the same value for $v'(1)$, the first-order Taylor approximation will be the same.

Problem 4 [30 points]
Let r denote the Euclidean (a.k.a Pythagorian) metric on $\mathbb{R}^n \times \mathbb{R}^n$ and let $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be defined by, for some $\alpha \in \mathbb{R}_{++}$ and all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

$$d(\mathbf{x}, \mathbf{y}) = \begin{cases} \alpha + r(\mathbf{x}, \mathbf{y}) & \text{if } \mathbf{x} \neq \mathbf{y} \\ 0 & \text{otherwise} \end{cases}.$$

Let ρ denote an arbitrary metric on $\mathbb{R}^m$. Now define the functions $\psi^+ : \mathbb{R}^{n+m} \times \mathbb{R}^{n+m} \to \mathbb{R}$ and $\psi^- : \mathbb{R}^{n+m} \times \mathbb{R}^{n+m} \to \mathbb{R}$ by, for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{n+m}$

$$\begin{aligned} \psi^+(\mathbf{x}, \mathbf{y}) &= d\big((x_1, ..., x_n), (y_1, ..., y_n)\big) + \rho\big((x_{n+1}, ..., x_{n+m}), (y_{n+1}, ..., y_{n+m})\big) \\ \psi^-(\mathbf{x}, \mathbf{y}) &= \big|d\big((x_1, ..., x_n), (y_1, ..., y_n)\big) - \rho\big((x_{n+1}, ..., x_{n+m}), (y_{n+1}, ..., y_{n+m})\big)\big| \end{aligned}$$

A) [*7 points*] Prove that ψ^+ is a metric on $\mathbb{R}^{n+m}$

We first need to observe that d is a metric on $\mathbb{R}^n$. It clearly satisfies $d(\mathbf{x}, \mathbf{y}) \geq 0$, $d(\mathbf{x}, \mathbf{y}) = 0$ iff $\mathbf{x} = \mathbf{y}$, and $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$. If, $\mathbf{x} = \mathbf{y}$, then d clearly satisfies the triangle inequality. Suppose then that $\mathbf{x} \neq \mathbf{y}$ and pick $\mathbf{z} \in \mathbb{R}^n$. Since $\mathbf{x} = \mathbf{z}$ implies $\mathbf{x} \neq \mathbf{y}$, it follows that

$$d(\mathbf{x}, \mathbf{y}) = \alpha + r(\mathbf{x}, \mathbf{y}) \leq \alpha + r(\mathbf{x}, \mathbf{z}) + r(\mathbf{z}, \mathbf{y}) \leq d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y})$$

The first inequality holds because r is a metric, and because either $\mathbf{x}$ and $\mathbf{z}$ or $\mathbf{z}$ and $\mathbf{y}$ (or both) are separated by at least α.
Now consider ψ^+. This clearly satisfies the first three requirements of a metric. We'll check that it satisfies the triangle inequality. Fix $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{n+m}$. There are two cases to consider. First suppose that there first n elements of $\mathbf{x}$ and $\mathbf{y}$ are identical. In this case,

$$\psi^+(\mathbf{x}, \mathbf{y}) = \rho\big((x_{n+1}, ..., x_{n+m}), (y_{n+1}, ..., y_{n+m})\big)$$

which, because ρ is a metric, is

$$\leq \rho\big((x_{n+1}, ..., x_{n+m}), (z_{n+1}, ..., z_{n+m})\big) + \rho\big((z_{n+1}, ..., z_{n+m}), (y_{n+1}, ..., y_{n+m})\big)$$

which, because d is a metric and hence nonnegative, is

$$\begin{aligned} \leq\ & d\big((x_1, ..., x_n), (z_1, ..., z_n)\big) + \rho\big((x_{n+1}, ..., x_{n+m}), (z_{n+1}, ..., z_{n+m})\big) \\ &+ d\big((z_1, ..., z_n), (y_1, ..., y_n)\big) + \rho\big((z_{n+1}, ..., z_{n+m}), (y_{n+1}, ..., y_{n+m})\big) \end{aligned}$$

which is, by definition

$$= \psi^+(\mathbf{x}, \mathbf{z}) + \psi^+(\mathbf{z}, \mathbf{y})$$

B) [*7 points*] Is ψ^- a metric on $\mathbb{R}^{n+m}$ for all possible specifications of r, ρ and α? If so, prove it, if not provide a counterexample. (Hint: in thinking about this problem, it is helpful to begin with the simplest of all metrics...)

It suffices to show that the triangle inequality fails for some $\mathbf{x}$ and $\mathbf{y}$ and $\mathbf{z}$ and some specification of r and ρ. Let r and ρ both be the discrete metric and let $m = n = 1$. Also, pick $\alpha < 1/2$. Let $\mathbf{x} = (0,0)$, $\mathbf{y} = (0,1)$ and $\mathbf{z} = (2,2)$. Now,

$$\psi^-(\mathbf{x},\mathbf{z}) \quad = \quad \psi^-(\mathbf{z},\mathbf{y}) \quad = \quad 1+\alpha-1 \quad = \quad \alpha$$

Hence,

$$\psi^-(\mathbf{x},\mathbf{z}) + \psi^-(\mathbf{z},\mathbf{y}) \quad = \quad 2\alpha \quad < \quad 1 \quad = \quad \psi^-(\mathbf{x},\mathbf{y})$$

This proves that ψ^- is not a metric in general.

C) [*7 points*] Write down a necessary condition for a sequence to be a convergent sequence in $\mathbb{R}^{n+m}$ with respect to ψ^+. Prove that it is necessary. (The following is *not* an acceptable answer: a necessary condition is that the first n components of the vectors converge w.r.t. d and the last m converge w.r.t. ρ.)

Consider $(\mathbf{x}^t) \in \mathbb{R}^{n+m}$ where for each n, $\mathbf{x}^t = (\mathbf{y}^t, \mathbf{z}^t)$, $\mathbf{y}^t \in \mathbb{R}^n$ and $\mathbf{z}^t \in \mathbb{R}^m$. A necessary condition for $(\mathbf{x}^t)$ to be a convergence sequence is that $(\mathbf{z}^t)$ is a convergent sequence in $\mathbb{R}^m$ w.r.t. ρ and there exists $\mathbf{x} \in \mathbb{R}^n$ and $N \in \mathbb{N}$ such that for $t > N$, $\mathbf{x}^t = \mathbf{x}$. You've seen the proof of this before.

D) [*9 points*] Write down a sufficient but not necessary condition for a sequence to be a convergent sequence in $\mathbb{R}^{n+m}$ with respect to ψ^+. Prove that it is sufficient, and show that it is not necessary. (The following is *not* an acceptable answer: a sufficient condition is that the first n components of the vectors converge w.r.t. d and the last m converge w.r.t. ρ.)

Consider $(\mathbf{x}^t) \in \mathbb{R}^{n+m}$ where for each n, $\mathbf{x}^t = (\mathbf{y}^t, \mathbf{z}^t)$, $\mathbf{y}^t \in \mathbb{R}^n$ and $\mathbf{z}^t \in \mathbb{R}^m$. A sufficient condition for $(\mathbf{x}^t)$ to be a convergent sequence is that $(\mathbf{z}^t)$ is a convergent sequence in $\mathbb{R}^m$ w.r.t. ρ and for all t, $\mathbf{x}^t = \mathbf{x}$. To see that this is not a necessary condition, let $n = m = 1$, let y^t be the sequence $\{0, 1, 1, 1...\}$ and z^t be the sequence $\{1, 1, 1, 1...\}$. The sequence (y^t, z^t) obviously converges to $(1,1)$, but does not satisfy the given condition.

Problem 5 [30 points]

Pick $a \in \mathbb{R}$ and let (x_n) be a sequence in $X \subset [-a, a]$, satisfying the following property: for all $n \in \mathbb{N}$, there exists $m > n$ such that $x_m \geq x_n$.

A) [*7 points*] Write down a nondecreasing subsequence of (x_n). You should define it inductively. (Hint: a useful fact is that every subset of $\mathbb{N}$ has a smallest element).

Let $\tau(1) = 1$; Now assume that $\tau(n)$ has been defined, and let $\tau(n+1) = \min\{m > n : x_m \geq x_n\}$. By assumption, $\tau(n+1)$ is well defined, and by construction τ is strictly increasing. Now, for each n, let $y_n = x_{\tau(n)}$. Since τ is strictly increasing, (y_n) is a subsequence of (x_n). Moreover, by construction, $y_{n+1} \geq y_n$, for all n.

B) [*23 points*] Suppose that (x_n) contains no convergent subsequence. Prove that the set X is not closed in $\mathbb{R}$.

(y_n) is a subsequence of (x_n) and is, moreover, nondecreasing. Moreover, since (y_n) is a sequence in X, it is bounded above by a. By the Axiom of Completeness, (y_n) converges to a point y in $\mathbb{R}$. However, by assumption (y_n) is not a convergent sequence. Therefore, y cannot belong to X. Since a closed set X contains all of it's accumulation points, and y is an accumulation point of X, X cannot be closed.