# FINAL EXAM

MONDAY DEC 14, '04

This is the final exam for ARE211. As announced earlier, this is an open-book exam. However, use of computers, calculators, Palm Pilots, cell phones, Blackberries and other comparable objects is forbidden.

If a question says "prove formally" then we mean it: a purely verbal answer is unlikely to be given full marks. However, if there's a step in your answer that involves a theorem that's given in the lecture notes, then you may state the theorem and reference the notes.

Allocate your 180 minutes in this exam wisely. The exam has 175 points, so aim for 1 minute per point. Make sure that you first do all the easy parts, before you move onto the hard parts. Always bear in mind that if you leave a part-question completely blank, you cannot conceivably get any marks for that part. The questions are designed so that, to some extent, even if you cannot answer some parts, you can still be able to answer later parts. So don't hesitate to leave a part out. You don't have to answer questions and parts of questions in the order that they appear on the exam, *provided that you clearly indicate the question/part-question you are answering.*

**Problem 1 (25 points).**

Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ be two concave functions and fix $\boldsymbol{\alpha} \in \mathbb{R}^2$. Let

$$Y = \{(y_1, y_2) \in \mathbb{R}^2 : \exists \mathbf{x}^1, \mathbf{x}^2 \in \mathbb{R}^2 \text{ s.t. } y_1 \leq f(\mathbf{x}^1), y_2 \leq g(\mathbf{x}^2) \text{ and } \mathbf{x}^1 + \mathbf{x}^2 \leq \boldsymbol{\alpha}\}.$$

To give you some intuition, functions $f$ and $g$ might be interpreted as production functions, $\mathbf{x}^1$ and $\mathbf{x}^2$ as input vectors and $Y$ as a production set.

A) (3 points) Give a formal definition of a convex set.

B) (3 points) Give a formal definition of a concave function.

C) (19 points) Using your answers to parts a) and b), prove formally that Y is convex.

**Problem 2 (25 points).**

Let $f : \mathbb{R}^n \to \mathbb{R}$ and fix $\mathbf{x} \in \mathbb{R}^n$.

A) (9 points) Suppose that at $\mathbf{x}$, all of the partial derivatives of $f$ exist.
   a) Give a mathematical definition of a partial derivative.
   b) Explain intuitively what this definition means in terms of the graph of $f$.
   c) Can you say that f is differentiable at $\mathbf{x}$?
   d) Does your answer to the previous question change if you are told that $f$ has directional derivatives in all directions at $\mathbf{x}$?

B) (16 points) Now suppose that $f$ is differentiable at $\mathbf{x}$. and denote by $df^{\mathbf{x}}$ the differential of $f$ at $\mathbf{x}$.
   a) Does $f$ have directional derivatives in all directions at $\mathbf{x}$?
   b) If so, for $h \in \mathbb{R}^n$, specify the directional derivative of $f$ at $\mathbf{x}$ in the direction $h$ in terms of the differential.
   c) Can you say that the partial derivatives of $f$, if they exist, are continuous at $\mathbf{x}$?
   d) Can you say that the partial derivatives of $f$ at $\mathbf{x}$, i.e., the $\frac{\partial f(\mathbf{x})}{\partial x_i}$'s, are continuous?

**Problem 3 (25 points).**

A) (9 points) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(\mathbf{x}) = \begin{bmatrix} x_1^{2/3} x_2^{1/3} \\ x_1^{1/3} x_2^{2/3} \end{bmatrix}$. Use the differential to estimate the value of $f(\cdot)$ at $(997, 29)$.

B) (8 points) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^4$. Use a second-order Taylor approximation of $f$ about $x = 2$ to estimate the value of $f(3)$.

C) (8 points) At what point must one compute $f'''(\cdot)$ so that the second order Taylor expansion of $f$ about $x = 2$ (including the remainder term) delivers exactly the difference between $f(3)$ and $f(2)$?

**Problem 4 (50 points).**

Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ be two differentiable functions. Consider the following NPP problem, which we shall refer to as (4$*$).

$$\max_{\mathbf{x}\geq 0} f(\mathbf{x}) \text{ s.t. } g(\mathbf{x}) \leq 0 \tag{4$*$}$$

where

(C0) $g$ is a quasi-convex function.

A) (4 points). Draw a constraint set that is consistent with assumption (C0).

B) (12 points). Assume in this part that $f$ is *not* quasi-concave.

(a) A *local solution* to (4$*$) is a point $\mathbf{x} \in \mathbb{R}^2$ such that for some neighborhood $U$ of $\mathbf{x}$, $f(\mathbf{x}) \geq f(\mathbf{x}')$, for all $\mathbf{x} \neq \mathbf{x}' \in U$ such that $g(\mathbf{x}') \leq 0$. Say that a local solution is a *strict* local solution if $f(\mathbf{x}) > f(\mathbf{x}')$, for all $\mathbf{x} \neq \mathbf{x}' \in U$ such that $g(\mathbf{x}') \leq 0$.

Show *graphically* that a point $\mathbf{x}$ may exist that is a strict local solution to (4$*$) but not a solution.

(b) Show *graphically* that a point $\mathbf{x}$ may exist that satisfies the KT conditions but is *not* a local solution to (4$*$).

C) (12 points). Fix $\mathbf{x} \in \mathbb{R}^2$ and suppose that

(C1) at $\mathbf{x}$ the KT conditions are satisfied.

(C2) $f$ is quasi-concave

(a) Give a condition of $\nabla f(\cdot)$ which,

- differs from M.K.9
- together with (C1) and (C2), guarantees that $\mathbf{x}$ is a solution to (4$*$).

Denote this condition by (C3).

(b) Provide a graphical example demonstrating that if (C1) and (C2) are satisfied but (C3) is not, then $\mathbf{x}$ may fail to be a solution to (4$*$).

D) (12 points). An alternative to conditions (C2) and (C3) is the one referred to in the lecture notes as condition M.K.9. If $f$ satisfies M.K.9 and $\mathbf{x}$ satisfies (C1), then $f$ is a solution to (4$*$).

(a) write down the assumption M.K.9

(b) explain why it is preferable to assume M.K.9 than to assume both (C2) and (C3). Your explanation should include an example. If you prefer, your example can be constructed using functions $f$ and $g$ that map $\mathbb{R}$ to $\mathbb{R}$.

E) (10 points). Prove formally that if $f$ is a concave function, then $f$ satisfies M.K.9.

**Hint:** Use the fact that $f$ is differentiable.

**Problem 5 (50 points).**

Fix $\alpha > 0$ and define $h(\alpha; x, y) = e^{\alpha(x+y)} + x^2 + y^4$. Now consider the unconstrained maximization problem

$$\text{minimize } h(\alpha; \cdot, \cdot) \text{ on } \mathbb{R}^2 \tag{5$*$}$$

For your convenience, Fig. 1 on the last page of the exam plots $z = e^\theta$. Also recall that $\frac{de^\theta}{d\theta} = e^\theta$.

A) (6 points). Compute $h(\alpha; 0, 0)$.

B) (6 points). Use your answer to part A) to prove that the solution to (5$*$) is the same as the following *constrained* minimization problem: minimize $h(\alpha; \cdot, \cdot)$ such that $x \in [-1, 1]$ and $y \in [-1, 1]$.

C) (6 points). Use your answer to part B) to prove that (5$*$) has a solution. Here and later in the question, you may take it as given that $h$ is a continuously differentiable function on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}_{++}$.

D) (6 points). For arbitrary $x, y \in \mathbb{R}^2$, compute the Hessian of $h(\alpha; \cdot, \cdot)$ at $(x, y)$.

E) (6 points). Use your answers to C) and D) to prove that (5$*$) has a *unique* solution.

F) (6 points).
   a) Write down the first order conditions for (5$*$).
   b) Prove that a solution to these first order conditions exist and is unique.
   c) Show that the unique solution to the FOC can be written as the level set corresponding to 0 of a function from $\mathbb{R}^3$ to $\mathbb{R}^2$ (to be denoted by $f$).

G) (6 points). Show that for all $\alpha > 0$, you can apply the implicit function theorem to the function $f$ given in Fc).

H) (8 points). Given $\bar{\alpha} > 0$, let $(x^*(\bar{\alpha}), y^*(\bar{\alpha}))$ denote the solution to (5$*$). Use Cramer's Rule to find an expression for $\frac{dx^*(\bar{\alpha})}{d\alpha}$.

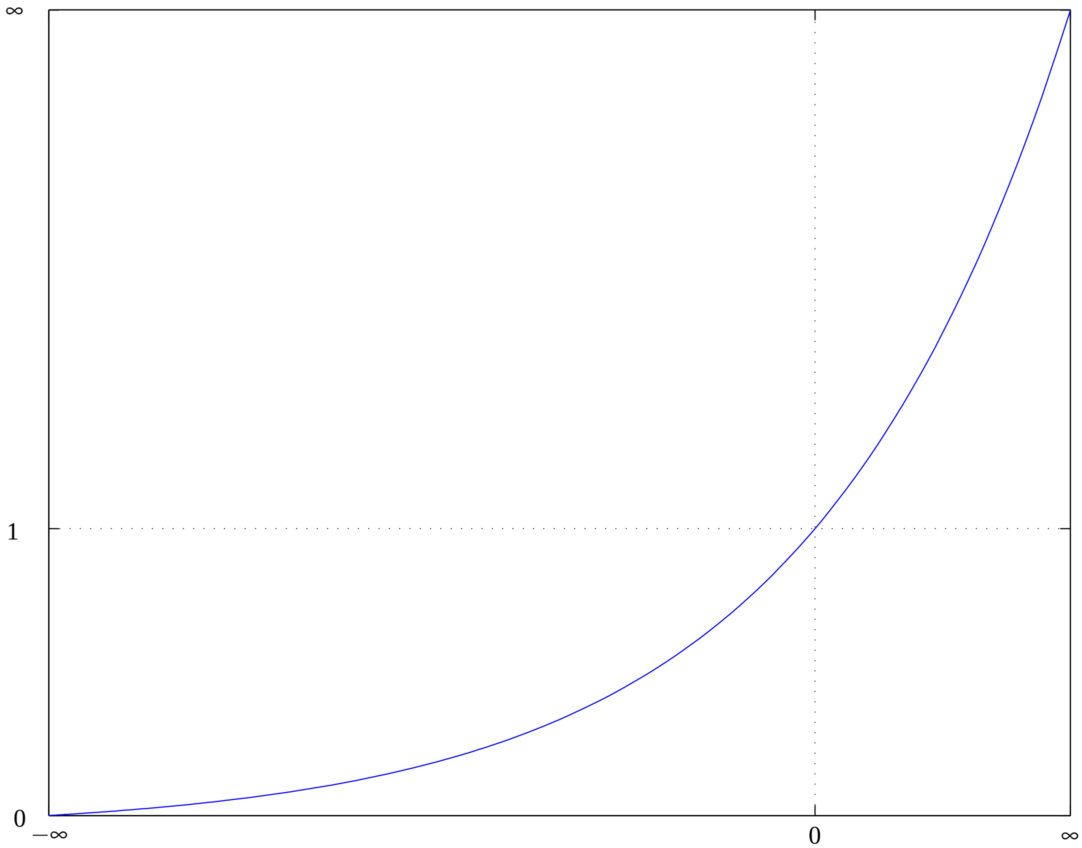


FIGURE 1. The graph of $z = e^{\theta}$