FINAL EXAM - ANSWER KEY

Problem 1 (25 points).

Let $f:\mathbb{R}^2 \to \mathbb{R}$ and $g:\mathbb{R}^2 \to \mathbb{R}$ be two concave functions and fix $\boldsymbol{\alpha} \in \mathbb{R}^2$. Let

$$Y = \{(y_1,y_2) \in \mathbb{R}^2 : \exists \mathbf{x}^1, \mathbf{x}^2 \in \mathbb{R}^2 \text{ s.t. } y_1 \leq f(\mathbf{x}^1), y_2 \leq g(\mathbf{x}^2) \text{ and } \mathbf{x}^1 + \mathbf{x}^2 \leq \boldsymbol{\alpha}\}.$$

To give you some intuition, functions f and g might be interpreted as production functions, $\mathbf{x}^1$ and $\mathbf{x}^2$ as input vectors and Y as a production set.

A) (3 points) Give a formal definition of a convex set.

A set S is convex if $\forall x,y \in S$ and all $\lambda \in [0,1]$, $\lambda x + (1-\lambda)y \in S$.

B) (3 points) Give a formal definition of a concave function.

A function $f : X \to Y$ is concave if the set of points below the graph of the function, i.e., $\{(x,y) \in X \times Y : y \leq f(x)\}$, is a convex set.

C) (19 points) Using your answers to parts a) and b), prove formally that Y is convex.

Pick $(\mathbf{y}^1, \mathbf{y}^2) \in Y$ and $\theta \in [0,1]$. We need to prove that $\mathbf{y}^\theta = (1-\theta)\mathbf{y}^1 + \theta\mathbf{y}^2 \in Y$. For $i = A, B$, since $\mathbf{y}^i \in Y$, there exists $\mathbf{x}^{i1}, \mathbf{x}^{i2} \in \mathbb{R}^2$ such that $y_1^i \leq f(\mathbf{x}^{i1})$, $y_2^i \leq g(\mathbf{x}^{i2})$ and $\mathbf{x}^{i1} + \mathbf{x}^{i2} \leq \boldsymbol{\alpha}$. For $j = 1,2$, define $\mathbf{x}^{\theta j} = (1-\theta)\mathbf{x}^{Aj} + \theta\mathbf{x}^{Bj}$. Since f and g are concave, we have, $f(\mathbf{x}^{\theta 1}) \geq (1-\theta)f(\mathbf{x}^{A1}) + \theta f(\mathbf{x}^{B1})$ and $g(\mathbf{x}^{\theta 2}) \geq (1-\theta)g(\mathbf{x}^{B2}) + \theta g(\mathbf{x}^{B2})$. Therefore

$$\begin{aligned}
\mathbf{y}_1^\theta &= (1-\theta)\mathbf{y}_1^A + \theta\mathbf{y}_1^B \leq (1-\theta)f(\mathbf{x}^{A1}) + \theta f(\mathbf{x}^{B1}) \leq f(\mathbf{x}^{\theta 1}) \\
\mathbf{y}_2^\theta &= (1-\theta)\mathbf{y}_2^A + \theta\mathbf{y}_2^B \leq (1-\theta)g(\mathbf{x}^{A2}) + \theta g(\mathbf{x}^{B2}) \leq g(\mathbf{x}^{\theta 2})
\end{aligned}$$

Clearly, $\mathbf{x}^{\theta 1} + \mathbf{x}^{\theta 2} \leq \boldsymbol{\alpha}$. Therefore, $\mathbf{x}^{\theta 1}$ and $\mathbf{x}^{\theta 2}$ satisfy the conditions required for $y^\theta = (y_1^\theta, y_2^\theta)$ to belong to Y.

Problem 2 (25 points).

Let $f:\mathbb{R}^n \to \mathbb{R}$ and fix $\mathbf{x} \in \mathbb{R}^n$.

A) (9 points) Suppose that at $\mathbf{x}$, all of the partial derivatives of f exist.

a) Give a mathematical definition of a partial derivative.

For $i = 1,\dots,n$, let $e^i \in \mathbb{R}^n$ be defined by, for $j = 1,\dots,n$: $e^i_j = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}$. The i'th partial derivative of f at $\mathbf{x}_0$ is now defined by

$$\frac{\partial f(\mathbf{x}_0)}{\partial x_i} = \lim_{|k|\to\infty} \frac{\left(f(\mathbf{x}_0 + \mathbf{e^i}/k) - f(\mathbf{x}_0)\right)}{1/k}$$

b) Explain intuitively what this definition means in terms of the graph of f.

It's the slope at $\mathbf{x}_0$ of the cross-section of the graph of f obtained by "slicing" the graph through the point vertically above $\mathbf{x}_0$ and in the direction of e^i.

c) Can you say that f is differentiable at $\mathbf{x}$?

No, existence of partials does not imply differentiability. See for example Figure 3 in the lecture notes CALCULUS2.

d) Does your answer to the previous question change if you are told that f has directional derivatives in all directions at $\mathbf{x}$?

No, existence of all directional derivatives does not imply differentiability. An example was provided in §4.3.7 of the lecture notes CALCULUS2

B) (16 points) Now suppose that f is differentiable at $\mathbf{x}$. and denote by $df^{\mathbf{x}}$ the differential of f at $\mathbf{x}$.

a) Does f have directional derivatives in all directions at $\mathbf{x}$?

Yes, differentiability implies the existence of all directional derivatives.

b) If so, for $h \in \mathbb{R}^n$, specify the directional derivative of f at $\mathbf{x}$ in the direction h in terms of the differential.

$\nabla f(\mathbf{x}) \cdot h/||h||$.

c) Can you say that the partial derivatives of f, if they exist, are continuous at $\mathbf{x}$?

No. Existence of continuous partials is a sufficient but not necessary condition for differentiability.

d) Can you say that the partial derivatives of f at $\mathbf{x}$, i.e., the $\frac{\partial f(\mathbf{x})}{\partial x_i}$'s, are continuous?

No. $\frac{\partial f(\mathbf{x})}{\partial x_i}$ is a point, and it doesn't make sense to say that a point is continuous.

Problem 3 (25 points).

A) (9 points) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(\mathbf{x}) = \begin{bmatrix} x_1^{2/3} x_2^{1/3} \\ x_1^{1/3} x_2^{2/3} \end{bmatrix}$. Use the differential to estimate the value of $f(\cdot)$ at $(997, 29)$.

$Jf(\mathbf{x}) = \begin{bmatrix} 2/3x_1^{-1/3}x_2^{1/3} & 1/3x_1^{2/3}x_2^{-2/3} \\ 1/3x_1^{-2/3}x_2^{1/3} & 2/3x_1^{2/3}x_2^{-1/3} \end{bmatrix}$. Now $(997,29) = (1000,27) + (-3,2)$ so we'll evaluate the differential of f at $\mathbf{x} = (1000,27)$ at the point $\mathbf{dx} = (-3,2)$. Now

$$f(\mathbf{x}) + Jf(\mathbf{x})\mathbf{dx} = \begin{bmatrix} 300 \\ 90 \end{bmatrix} + \begin{bmatrix} \frac{1}{5} & \frac{100}{27} \\ \frac{3}{100} & \frac{20}{9} \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 300 + \frac{200}{27} - \frac{3}{5} \\ 90 + \frac{40}{9} - \frac{9}{100} \end{bmatrix} = \begin{bmatrix} 306.8074 \\ 94.3544 \end{bmatrix}$$

So that the

B) (8 points) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^4$. Use a second-order Taylor approximation of f about $x = 2$ to estimate the value of $f(3)$.

The second order Taylor expansion of f about x is $4x^3dx + 6x^2dx^2$. Evaluating at $x = 2$ and $dx = 3 - 2 = 1$, we obtain $T_2(f,2,1) = 56$. So the second order Taylor approximation to $f(3)$ is $f(2) + 56 = 72$.

C) (8 points) At what point must one compute $f'''(\cdot)$ so that the second order Taylor expansion of f about $x = 2$ (including the remainder term) delivers exactly the difference between $f(3)$ and $f(2)$?

Since $f(3) = 81$, the second order Taylor approximation to $f(3)$ is 9 short of its target. We need to make this up by choosing y so that the third order Taylor term $Tf_3(y,dx) = 4ydx = 9$. Since $dx = 1$, we have $y = 9/4$. That is, we need to evaluate $f'''(\cdot)$ at $9/4$.

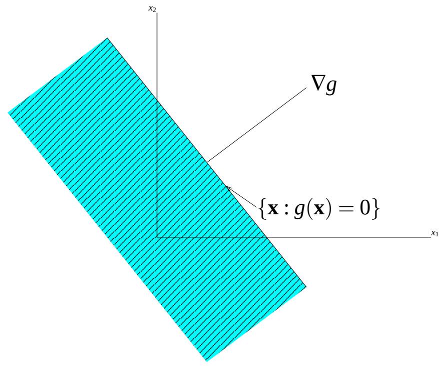

FIGURE 1. Quasi-convex constraint set

Problem 4 (50 points).

Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ be two differentiable functions. Consider the following NPP problem, which we shall refer to as (4∗).

$$\max_{\mathbf{x} \geq 0} f(\mathbf{x}) \text{ s.t. } g(\mathbf{x}) \leq 0 \tag{4$*$}$$

where

(C0) g is a quasi-convex function.

A) (4 points). Draw a constraint set that is consistent with assumption (C0).

See Fig. 1

B) (12 points). Assume in this part that f is *not* quasi-concave.

(a) A *local solution* to (4∗) is a point $\mathbf{x} \in \mathbb{R}^2$ such that for some neighborhood U of $\mathbf{x}$, $f(\mathbf{x}) \geq f(\mathbf{x}')$, for all $\mathbf{x} \neq \mathbf{x}' \in U$ such that $g(\mathbf{x}') \leq 0$. Say that a local solution is a *strict* local solution if $f(\mathbf{x}) > f(\mathbf{x}')$, for all $\mathbf{x} \neq \mathbf{x}' \in U$ such that $g(\mathbf{x}') \leq 0$.

Show *graphically* that a point $\mathbf{x}$ may exist that is a strict local solution to (4∗) but not a solution.

See Fig. 2. $\mathbf{x}$ is a local solution, but $\mathbf{x}'$ belongs to the feasible set and $f(\mathbf{x}') > f(\mathbf{x})$.

(b) Show *graphically* that a point $\mathbf{x}$ may exist that satisfies the KT conditions but is *not* a local solution to (4∗).

See Fig. 3.

C) (12 points). Fix $\mathbf{x} \in \mathbb{R}^2$ and suppose that

(C1) at $\mathbf{x}$ the KT conditions are satisfied.

(C2) f is quasi-concave

(a) Give a condition of $\nabla f(\cdot)$ which,

- differs from M.K.9

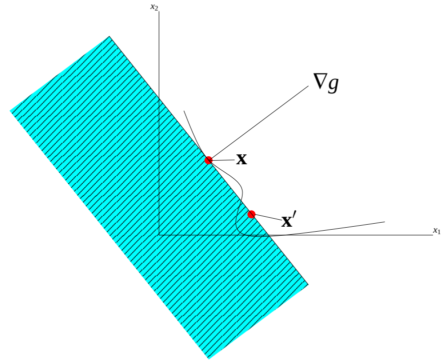

FIGURE 2. A local solution that is not a solution

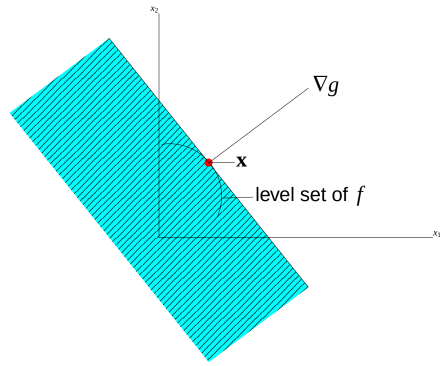

FIGURE 3. A point satisfying KT that is not a local solution

- together with (C1) and (C2), guarantees that $\mathbf{x}$ is a solution to (4$*$).

Denote this condition by (C3).

(C3) The gradient of f never vanishes. That is, for all $\mathbf{x}$ in the domain of f, $\nabla f(\mathbf{x}) \neq 0$

(b) Provide a graphical example demonstrating that if (C1) and (C2) are satisfied but (C3) is not, then $\mathbf{x}$ may fail to be a solution to (4$*$). If you prefer, your example can be constructed using functions f and g that map $\mathbb{R}$ to $\mathbb{R}$.

See Fig. 4. The gradient of f vanishes at $\mathbf{x}$, the KT conditions are satisfied but f is not maximized on the constraint set at this point.

D) (12 points). An alternative to conditions (C2) and (C3) is the one referred to in the lecture notes as condition M.K.9. If f satisfies M.K.9 and $\mathbf{x}$ satisfies (C1), then f is a solution to (4$*$).

(a) write down the assumption M.K.9

$$\forall \mathbf{x}, \mathbf{x}' \in X, \text{ if } f(\mathbf{x}') > f(\mathbf{x}) \text{ then } \nabla f(\mathbf{x}) \cdot (\mathbf{x}' - \mathbf{x}) > 0.$$

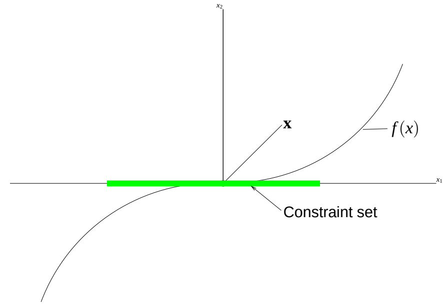

FIGURE 4. A point satisfying (C1) and (C2) that is not a solution

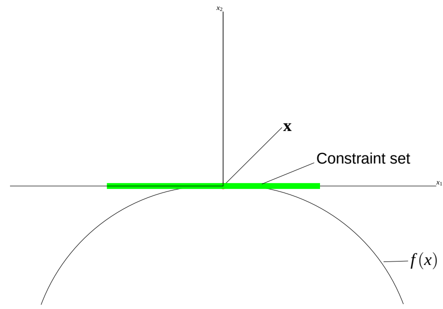

FIGURE 5. A point satisfying M.K.9 but not (C3)

(b) explain why it is preferable to assume M.K.9 than to assume both (C2) and (C3). Your explanation should include an example. If you prefer, your example can be constructed using functions f and g that map $\mathbb{R}$ to $\mathbb{R}$.

In general, one wants to impose the weakest possible assumption which accomplishes the task at hand. M.K.9 is much weaker than (C3), and hence preferable on general grounds. More specifically, if (C3) is imposed, then one cannot obtain an *unconstrained* solution to the NPP. Fig. 5 provides an example of an NPP that one would not want to exclude.

E) (10 points). Prove formally that if f is a concave function, then f satisfies M.K.9.

Hint: Use the fact that f is differentiable.

If f is concave, then for all $\mathbf{x}, \mathbf{x}'$, $\nabla f(\mathbf{x})(\mathbf{x}' - \mathbf{x}) \geq f(\mathbf{x}') - f(\mathbf{x})$. Therefore, if $f(\mathbf{x}') - f(\mathbf{x}) > 0$, then $\nabla f(\mathbf{x})(\mathbf{x}' - \mathbf{x}) \geq f(\mathbf{x}') - f(\mathbf{x}) > 0$. Hence M.K.9 is satisfied.

Problem 5 (50 points).

Fix $\alpha > 0$ and define $h(\alpha; x, y) = e^{\alpha(x+y)} + x^2 + y^4$. Now consider the unconstrained maximization problem

$$\text{minimize } h(\alpha; \cdot, \cdot) \text{ on } \mathbb{R}^2 \tag{5$*$}$$

For your convenience, Fig. 6 on the last page of the exam plots $z = e^\theta$. Also recall that $\frac{de^\theta}{d\theta} = e^\theta$.

A) (6 points). Compute $h(\alpha; 0, 0)$.

$$h(\alpha; 0, 0) \quad = \quad e^{\alpha 0} + 0^2 + 0^4 \quad = \quad e^0 \quad = \quad 1$$

B) (6 points). Use your answer to part A) to prove that the solution to (5$*$) is the same as the following *constrained* minimization problem: minimize $h(\alpha; \cdot, \cdot)$ such that $x \in [-1, 1]$ and $y \in [-1, 1]$.

If either $|x| > 1$ or $|y| > 1$, then either x^2 or y^4 exceeds 1, while the other is positive. Moreover, $h(\alpha; \cdot, \cdot)$ is nonnegative function. Hence,

$$h(\alpha; x, y) \quad > \quad e^{\alpha(x+y)} + 1 \quad > \quad 1 \quad = \quad h(\alpha; 0, 0)$$

. Conclude that a necessary condition for $h(\alpha; \cdot, \cdot)$ to be minimized at (x, y) is that $\max(|x|, |y|) \leq 1$.

C) (6 points). Use your answer to part B) to prove that (5$*$) has a solution. Here and later in the question, you may take it as given that h is a continuously differentiable function on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}_{++}$.

Since $h(\alpha; \cdot, \cdot)$ is continuous, and the set $[-1, 1]^2$ is a compact set, the Weierstrass theorem implies that $h(\alpha; \cdot, \cdot)$ attains a minimum on this set. Part B) then implies that this minimum is in fact an *unconstrained* minimum for $h(\alpha; \cdot, \cdot)$. This minimum is the solution to (5$*$).

D) (6 points). For arbitrary $x, y \in \mathbb{R}^2$, compute the Hessian of $h(\alpha; \cdot, \cdot)$ at (x, y).

$$Hh(\alpha; x, y) \quad = \quad \begin{bmatrix} \alpha^2 e^{\alpha(x+y)} + 2 & \alpha^2 e^{\alpha(x+y)} \\ \alpha^2 e^{\alpha(x+y)} & \alpha^2 e^{\alpha(x+y)} + 12y^2 \end{bmatrix}$$

E) (6 points). Use your answers to C) and D) to prove that (5$*$) has a *unique* solution.

The first principal minor of $Hh(\alpha;x,y)$ is $\alpha^2 e^{\alpha(x+y)}+2$ which is positive. The second principal minor is

$$\det(Hh_{(x,y)}(\alpha;x,y)) \quad = \quad 24y^2+(2+12y^2)\alpha^2 e^{\alpha(x+y)} \quad > \quad 0$$

This establishes that $h(\alpha,\cdot,\cdot)$ is strictly convex. A strictly convex function has at most one global minimum. Combining this with C), we have established that $(5*)$ has a unique solution.

F) (6 points).

a) Write down the first order conditions for $(5*)$.

The first order conditions are

$$\begin{aligned} \frac{\partial h(\alpha;x,y)}{\partial x} &= \alpha e^{\alpha(x+y)} + 2x &= 0 \\ \frac{\partial h(\alpha;x,y)}{\partial y} &= \alpha e^{\alpha(x+y)} + 4y^3 &= 0 \end{aligned}$$

b) Prove that a solution to these first order conditions exist and is unique.

For a strictly convex function, the first order conditions are necessary and sufficient for a unique solution. Since we've proved in C) that $(5*)$ has a solution, it follows that there exists a unique pair (x,y) which solves the equation system in F).

c) Show that the unique solution to the FOC can be written as the level set corresponding to 0 of a function from $\mathbb{R}^3$ to $\mathbb{R}^2$ (to be denoted by f).

Define the function $f:\mathbb{R}^3 \to \mathbb{R}^2$, defined by

$$f(\alpha,x,y) \quad = \quad \begin{bmatrix} \alpha e^{\alpha(x+y)} + 2x \\ \alpha e^{\alpha(x+y)} + 4y^3 \end{bmatrix}$$

For each $\alpha > 0$, the FOC specify that $f(\alpha,x,y)$ must equal zero. Hence these conditions can be written as the level set corresponding to 0 of f.

G) (6 points). Show that for all $\alpha > 0$, you can apply the implicit function theorem to the function f given in Fc).

Consider the function $f(\alpha,\cdot,\cdot)$. Clearly, $Jf_{(x,y)}(\alpha,\cdot,\cdot)$ is the Hessian $Hh_{(x,y)}$ specified above. We've already checked that this matrix has a positive determinant. Hence the condition of the implicit function is satisfied.

H) (8 points). Given $\bar{\alpha} > 0$, let $(x^*(\bar{\alpha}), y^*(\bar{\alpha}))$ denote the solution to $(5*)$. Use Cramer's Rule to find an expression for $\frac{dx^*(\bar{\alpha})}{d\alpha}$.

Plugging through the implicit function theorem, we obtain

$$\frac{dx^*(\bar{\alpha})}{d\alpha} = \frac{-12y^*(\bar{\alpha})^2\bar{\alpha}e^{\bar{\alpha}(x^*(\bar{\alpha})+y^*(\bar{\alpha}))}}{24y^*(\bar{\alpha})^2+(2+12y^*(\bar{\alpha})^2)\bar{\alpha}^2e^{\bar{\alpha}(x^*(\bar{\alpha})+y^*(\bar{\alpha}))}}\left(1+\bar{\alpha}(x^*(\bar{\alpha})+y^*(\bar{\alpha}))\right)$$

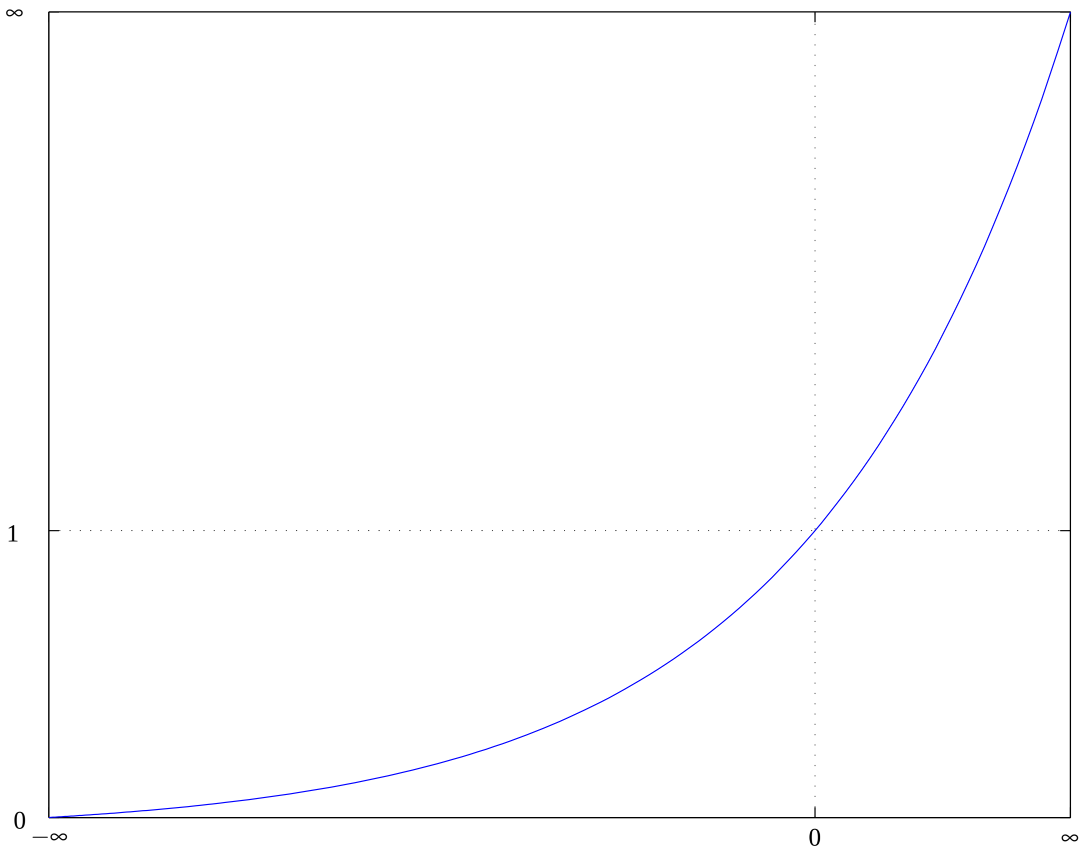

FIGURE 6. The graph of $z = e^{\theta}$